1	2	3	Total
/25	/45	/30	/100

Economics 172A Midterm Exam NAME________________________________

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Your grade from this exam is 35% of your course grade. The exam ends promptly at 9:20, so you have 80 minutes. You may not use books, notes, calculators or other electronic devices. (Calculators shouldn't be needed.) There are three questions, weighted as indicated. Answer them all. If you cannot give a complete answer, try to explain what you understand about the answer. Write your name in the space above, now. Write your answers below the questions, on the back of the page, or on separate sheets. Explain your arguments and show your work. Good luck!

1. (25 points) Comfortable Hands is a company which features a product line of winter gloves for the entire family — men, women, and children. They are trying to decide what mix of these three types of gloves to produce. Comfortable Hands' manufacturing labor force is unionized. Each full-time employee works a 40-hour week. In addition, by union contract, the number of full-time employees can never drop below 20. Nonunion, part-time workers can also be hired with the following union-imposed restrictions: (1) each part-time worker works 20 hours per week, and (2) there must be at least 2 full-time employees for each part-time employee.

All three types of gloves are made out of the same 100% genuine cowhide leather. Comfortable Hands has a long term contract with a supplier of the leather, and receives a 5,000 square feet shipment of the material each week. The material requirements and labor requirements, along with the *gross profit* per glove sold (not considering labor costs) is given in the following table.

Glove	Material Required (square feet)	Labor Required (minutes)	Gross Profit (per pair)
Men's	2	30	\$8
Women's	1.5	45	\$10
Children's	1	40	\$6

Each full-time employee earns \$13 per hour, while each part-time employee earns \$10 per hour. Management wishes to know what mix of each of the three types of gloves to produce per week, as well as how many full-time and how many part-time workers to employ. They would like to maximize their *net profit* — their gross profit from sales minus their labor costs.

Letting M = number of men's gloves to produce per week, W = number of women's gloves to produce per week, C = number of children's gloves to produce per week, F = number of full-time workers to employ, and P = number of part-time workers to employ, formulate a linear programming model for this problem, and put it into standard form (with only $\leq$ constraints).

[Choose M, W, C, F, P to maximize $8M + 10W + 6C - 13(40)F - 10(20)P$,

subject to

$$2M + 1.5W + C \leq 5000$$
$$30M + 45W + 40C \leq 40(60)F + 20(60)P$$
$$F \geq 20 \text{ or } -F \leq -20$$
$$F \geq 2P \text{ or } -F + 2P \leq 0$$
$$M \geq 0,\ W \geq 0,\ C \geq 0,\ F \geq 0,\ P \geq 0.]$$

2. (45 points)

(a) Put the following minimization problem in standard form (with only ≥ constraints):

Choose x_1 and x_2 to minimize $3x_1 + 2x_2$ subject to

$$2x_1 + x_2 \geq 10$$
$$-3x_1 + 2x_2 \leq 6$$
$$x_1 + x_2 \geq 6$$
$$x_1 \geq 0, \quad x_2 \geq 0.$$

[Flipping the ≤ constraint to ≥:
Choose x_1 and x_2 to minimize $3x_1 + 2x_2$ subject to

$$2x_1 + x_2 \geq 10$$
$$3x_1 - 2x_2 \geq -6$$
$$x_1 + x_2 \geq 6$$
$$x_1 \geq 0, \quad x_2 \geq 0.]$$

(b) After putting the problem in (a) in standard form, construct its dual.

[Choose y_1, y_2, and y_3 to maximize $10y_1 - 6y_2 + 6y_3$ subject to

$$2y_1 + 3y_2 + y_3 \leq 3$$
$$y_1 - 2y_2 + y_3 \leq 2$$
$$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0.$$

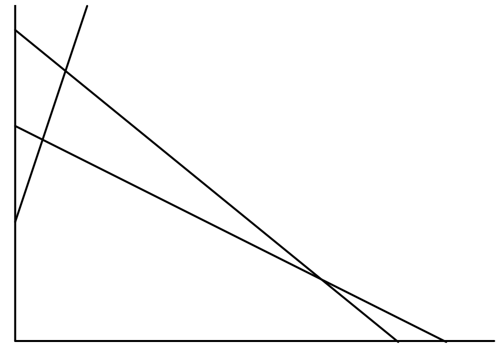
(Since the primal is a minimization problem, the dual is a maximization problem. Just as the primal constraints are in standard ≥ form for a minimization problem, the dual constraints are in standard ≤ form for a maximization problem. There is a dual control variable/shadow price for each primal constraint, and the dual objective function coefficients are the corresponding constraint constants from the primal. The dual constraint constants are the primal objective function coefficients. Finally, the coefficients of the dual constraints are obtained from those of the primal constraints by transposing the matrix.)

(c) Verify that the dual has a nonempty feasible region. In this case, what does the Duality Theorem say about the possibilities for the dual and the primal having nonempty feasible regions, having unbounded objective function values, and/or having solutions?

[$y_1 = y_2 = y_3 = 0$ satisfies all the dual constraints. By the Duality Theorem, either the primal is infeasible and the dual has an unbounded objective function value, or both the dual and the primal have solutions.]

(d) Putting x_2 on the vertical axis and x_1 on the horizontal axis, graph the feasible region of the transformed primal. Is it nonempty? Is it unbounded?

[The feasible region is the area northeast of both x_1 and x_2 axes and to the east or northeast of the other three lines in the figure (not drawn to scale). Thus it's nonempty but unbounded.]



(e) Use your graph to show that the transformed primal has a solution. (Don't forget that it's a minimization problem.) Calculate the solution.
[The objective function contours have equations $3x_1 + 2x_2$ = constant. The slopes of these contours are between the slopes of the two constraints whose boundaries have negative slopes in the figure (whose boundary equations are $2x_1 + x_2 = 10$ and $x_1 + x_2 = 6$, with slopes bracketing that of $3x_1 + 2x_2$ = constant). Thus there is a solution even though the feasible region is unbounded. It occurs at the intersection of these two constraints: (solving their equations) $x_1^* = 4$ and $x_2^* = 2$. (This satisfies the third constraint: $3x_1^* - 2x_2^* = 8 \geq -6$.)

(f) Use complementary slackness to use your solution for the primal to find a solution for the dual.
[Because both x_1^* and $x_2^* > 0$, both constraints in the dual must be binding. Because the solution of the primal leaves the second constraint slack, $y_2^* = 0$. Plugging this into $2y_1 + 3y_2 + y_3 = 3$ and $y_1 - 2y_2 + y_3 = 2$ and solving simultaneously, $2y_1 + y_3 = 3$ and $y_1 + y_3 = 2$, so $y_1^* = y_3^* = 1$. Checking, $y_1^* = y_3^* = 1$ and $y_2^* = 0$ satisfy all the dual constraints.]

(g) Use your solution for the dual to estimate the value in the primal (the change in the primal objective function value) of increasing the constraint constant of the first constraint, 10, to 11. Use your solution for the dual to estimate the value in the primal of increasing the constraint constant of the second constraint, -6, to -5. Do your conclusions accord with intuition? Explain.
[Raising the 10 to an 11 will increase the objection function value by approximately the shadow price of the first constraint, $y_1^* = 1$. This is intuitive because the change makes the feasible region smaller, so it should hurt (and hurting means higher objective function value, because it's a minimization problem). Raising the -6 to a -5 will increase the objection function value by approximately the shadow price of the second constraint, $y_2^* = 0$. This is intuitive because the change makes the feasible region smaller, but the constraint is slack anyway so it shouldn't hurt.]

3. (30 points) Ken and Larry, Inc., supplies its ice cream parlors with three flavors of ice cream: chocolate, vanilla, and banana. Due to extremely hot weather and a high demand for its products, the company has run short of its supply of ingredients: milk, sugar, and cream. Hence, they will not be able to fill all the orders received from their retail outlets, the ice cream parlors. Due to these circumstances, the company has decided to choose the amount of each flavor to produce that will maximize total profit, given the constraints on supply of the basic ingredients.

The chocolate, vanilla, and banana flavors generate, respectively, \$1.00, \$0.90, and \$0.95 of profit per gallon sold. The company has only 200 gallons of milk, 150 pounds of sugar, and 60 gallons of cream left in its inventory. The linear programming formulation for this problem is shown below in algebraic form.

Let C = gallons of chocolate ice cream produced,
V = gallons of vanilla ice cream produced,
B = gallons of banana ice cream produced.

Choose C, V, and B to maximize $1.00\,C + 0.90\,V + 0.95\,B$,

subject to
Milk: $0.45\,C + 0.50\,V + 0.40\,B \leq 200$ gallons
Sugar: $0.50\,C + 0.40\,V + 0.40\,B \leq 150$ pounds
Cream: $0.10\,C + 0.15\,V + 0.20\,B \leq 60$ gallons
$C \geq 0,\ V \geq 0,\ B \geq 0.$

This problem was solved using the Excel Solver. The spreadsheet (already solved) and the sensitivity report are shown below. [Note: The numbers in the sensitivity report for the milk constraint are missing on purpose, since you will be asked to fill in these numbers in part (f).]

	A	B	C	D	E	F	G
1		Resource Usage Per Unit of Each Activity					
2			Activity				Resource
3	Resource	Chocolate	Vanilla	Banana	Totals		Available
4	Milk	0.45	0.5	0.4	180	≤	200
5	Sugar	0.5	0.4	0.4	150	≤	150
6	Cream	0.1	0.15	0.2	60	≤	60
7	Unit Profit	1	0.9	0.95	$341.25		
8	Solution	0	300	75			

Changing Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B8	Solution Chocolate	0	-0.0375	1	0.0375	1E+30
C8	Solution Vanilla	300	0	0.9	0.05	0.0125
D8	Solution Banana	75	0	0.95	0.021429	0.05

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
E4	Milk Totals					
E5	Sugar Totals	150	1.875	150	10	30
E6	Cream Totals	60	1	60	15	3.75

For each of the following parts, answer the question as specifically and completely as is possible without solving the problem again on the Excel Solver. Note: Each part is independent (that is, any change made to the model in one part does not apply to any other parts).

(a) What is the optimal solution and total profit?
[The optimal solution is to produce no chocolate ice cream, 300 gallons of vanilla ice cream, and 75 gallons of banana ice cream. Total profit will be $341.25.]

(b) Suppose the profit per gallon of banana changes to $1.00. Will the optimal solution change, and what can be said about the effect on total profit?
[The optimal solution will change since $1.00 is outside the allowable range of $(0.95-0.05) to $(0.95 + 0.021429). The profit will go up, but how much can't be determined without re-solving.]

(c) Suppose the profit per gallon of banana changes to 92¢. Will the optimal solution change, and what can be said about the effect on total profit?
[The optimal solution will not change since $0.92 is within the allowable range. The total profit will decrease by (75)$0.03 = $2.25 to $339.]

(d) Suppose the company discovers that 3 gallons of cream have gone sour and so must be thrown out. Will the optimal solution change, and what can be said about the effect on total profit?

[The optimal solution will change. Since the change is within the allowable range (the allowable decrease is 3.75), we can calculate the change in profit using the shadow price (1*3 = 3). The new profit will be $338.25.]

(e) Suppose the company has the opportunity to buy an additional 15 pounds of sugar at a total cost of $15. Should they? Explain.
[This increase is outside of the allowable increase of 10, so the problem will have be re-solved to determine whether this is worthwhile.]

(f) Fill in all the sensitivity report information for the milk constraint, given just the optimal solution for the problem. Explain how you were able to deduce each number.
[The final value is 180 as shown in the totals column in the solution. The shadow price is 0 since we are using less milk than we have available. The right-hand side value is 200 as given in the problem. The allowable increase is infinity since we are already using less than is available. The allowable decrease is 20 since the solution will change once the right-hand side drops below 180.]