simplify presentation he first assumed that Σ is a σ-algebra, i.e., an algebra such that for every sequence of events $(A_i)_{i=1}^{\infty}$ it contains its union $\bigcup_{i=1}^{\infty} A_i$. He then required that $P(\bigcup_{i=1}^{\infty} A_i) = \Sigma_{i=1}^{\infty} P(A_i)$ if the A_i's are pairwise disjoint.

This last property is referred to as σ-additivity of the probability P. In this way Kolmogorov transformed large parts of probability theory into (a special case of) measure theory. Thus an expectation of a random variable X is

$$E(X) = \int_S X(s)\,\mathrm{d}P(s) \tag{5.5}$$

where the right side is a Lebesgue integral (if it exists...), defined as a limit of integrals of random variables with countably many values. Let Y be such a random variable with values $(y_i)_{i=1}^{\infty}$, then

$$E(Y) = \sum_{i=1}^{\infty} P(Y = y_i) y_i \tag{5.6}$$

if the right side is absolutely convergent.

5.7 *An example* will now be introduced of a finitely additive probability, i.e. a probability for which (5.3) holds but (5.6) does not hold. Let S be the set of rational numbers in the interval [0, 1] and let Σ be the algebra of all subsets of S. (It is in fact a σ-algebra.) For $0 \leqslant \alpha \leqslant \beta \leqslant 1$ define $P(S \cap [\alpha, \beta]) = \beta - \alpha$ and extend P to all subsets of Σ. For each s in S, $P(s) = 0$. Since S is countable we can write $S = \{s_1, s_2, \ldots\}$ and $1 = P(S) > \Sigma_{i=1}^{\infty} P(s_i) = 0$. Defining $Y(s_i) = 1/i$ for all i, we get a contradiction to (5.6). The finitely additive probability P has also the property implied by Savage's P6 (see 2.4): If $P(A) > 0$ then there is an event $B \subset A$ such that $0 < P(B) < P(A)$.

5.8 *Distributions*. A non-decreasing right continuous function on the extended real line is called a distribution function if $F(-\infty) = 0$ and $F(\infty) = 1$. Given a random variable X, its distribution function F_X is defined by $F_X(\alpha) = P(X \leqslant \alpha)$ for all real α. Then

$$E(X) = \int_0^{\infty} [1 - F_X(\alpha)]\,\mathrm{d}\alpha - \int_{-\infty}^{0} F(\alpha)\,\mathrm{d}\alpha \tag{5.9}$$

which is the dual of formula (5.2). If the distribution F_X is smooth we say that the random variable X has a density $f_X: R \to R$, which is the derivative of F_X. In this case

$$E(X) = \int_{-\infty}^{\infty} \alpha f(\alpha)\,\mathrm{d}\alpha \tag{5.10}$$

5.11 *Non-additive probability*. A function $P: \Sigma \to [0, 1]$ is said to be *non-additive* probability (or capacity) if $P(S) = 1$, $P(\phi) = 0$ and for $A \subset B, P(A) \leqslant P(B)$. Choquet (1954) suggested to integrate a random variable with respect to non-additive probability by formula (5.2).

DAVID SCHMEIDLER AND PETER WAKKER

See also ALLAIS PARADOX; MEAN VALUES; RISK; SUBJECTIVE PROBABILITY; UNCERTAINTY; UTILITY THEORY AND DECISION-MAKING

BIBLIOGRAPHY

Anscombe, F.J. and Aumann, R.J. 1963. A definition of subjective probability. *Annals of Mathematical Statistics* 34, 199–205.

Bernoulli, D. 1738. Specimen theoriae novae de mensura sortis. *Commentarii Academiae Scientiarum Imperialis Petropolitanae* 5, 175–92. Translated into English by L. Sommer (1954) as: Exposition of a new theory on the measurement of risk, *Econometrica* 12, 23–36; or in *Utility Theory: A Book of Readings*, ed. A.N. Page, New York: Wiley, 1986.

Choquet, G. 1953–54. Theory of capacities. *Annales de l'Institut Fourier* (Grenoble), 131–295.

de Finetti, B. 1931. Sul significato soggettivo della probabilita. *Fundamenta Mathematicae* 17, 298–329.

de Finetti, B. 1937. La prévision: ses lois logiques, ses sources subjectives. *Annales de l'Institut Henri Poincaré* 7, 1–68. Translated into English in *Studies in Subjective Probability*, ed. H.E. Kyburg and H.E. Smokler, 1964, New York: Wiley.

Ellsberg, D. 1961. Risk, ambiguity, and the Savage axioms. *Quarterly Journal of Economics* 75, 643–69.

Fishburn, P.C. 1970. *Utility Theory for Decision Making*. New York: Wiley.

Gilboa, I. 1986. Non-additive probability measures and their applications in expected utility theory. PhD Thesis submitted to Tel Aviv University.

Herstein, I.N. and Milnor, J. 1953. An axiomatic approach to measurable utility. *Econometrica* 21, 291–7.

Karni, E. 1985. *Decision-Making under Uncertainty: The Case of State-Dependent Preferences*. Cambridge, Mass.: Harvard University Press.

Kolmogorov, A.N. 1933. *Grundbegriffe der Wahrscheinlichkeitrechnung*. Berlin. Translated into English by Nathan Morrison (1950, 2nd edn, 1956), New York: Chelsea Publishing Company.

Loeve, M. 1963. *Probability Theory*. 3rd edn, Princeton: Van Nostrand.

von Neumann, J. and Morgenstern, O. 1947. *Theory of Games and Economic Behavior*. 2nd edn, Princeton: Princeton University Press.

Ramsey, F.P. 1931. Truth and probability. In *The Foundations of Mathematics and Other Logical Essays*, ed. R.B. Braithwaite, New York: Harcourt, Brace.

Savage, L.J. 1954. *The Foundations of Statistics*. New York: Wiley, 2nd edn, 1972.

Schmeidler, D. 1984. Subjective probability and expected utility without additivity. CARESS, University of Pennsylvania and IMA University of Minnesota, mimeo.

Wald, A. 1951. *Statistical Decision Functions*. New York: Wiley.

Wakker, P.P. 1986. Representations of choice situations. PhD thesis, University of Tilburg, Department of Economics.

expected utility hypothesis. The expected utility hypothesis of behaviour towards risk is essentially the hypothesis that the individual decision-maker possesses (or acts as if possessing) a 'von Neumann–Morgenstern utility function' $U(\cdot)$ or 'von Neumann–Morgenstern utility index' $\{U_i\}$ defined over some set of outcomes, and when faced with alternative risky prospects or 'lotteries' over these outcomes, will choose that prospect which maximizes the expected value of $U(\cdot)$ or $\{U_i\}$. Since the outcomes could represent alternative wealth levels, multidimensional commodity bundles, time streams of consumption, or even non-numerical consequences (e.g. a trip to Paris), this approach can be applied to a tremendous variety of situations, and most theoretical research in the economics of uncertainty, as well as virtually all applied work in the field (e.g. optimal trade, investment or search under uncertainty) is undertaken in the expected utility framework.

As a branch of modern consumer theory (e.g. Debreu, 1959, ch. 4), the expected utility model proceeds by specifying a set of objects of choice and assuming that the individual possesses a preference ordering over these objects which may be represented by a real-valued maximand or 'preference function' $V(\cdot)$, in the sense that one object is preferred to another if and only if it is assigned a higher value by this preference function. However, the expected utility model differs from the theory of choice over non-stochastic commodity bundles in two important respects. The first is that since it is a theory of choice under uncertainty, the objects of choice are not deterministic outcomes but rather probability distributions over these outcomes. The second difference is that, unlike in the non-stochastic case, the expected utility model imposes a very specific restriction on the functional form of the preference function $V(\cdot)$.

The formal representation of the objects of choice, and hence of the expected utility preference function, depends upon the structure of the set of possible outcomes. When there are a finite number of outcomes $\{x_1, \ldots, x_n\}$, we can represent any probability distribution over this set by its vector of probabilities $P = (p_1, \ldots, p_n)$ (where $p_i = \text{prob}(\tilde{x} = x_i)$), and the preference function takes the form

$$V(P) = V(p_1, \ldots, p_n) \equiv \Sigma U_i p_i.$$

When the outcome set consists of the real line or some subset of it, probability distributions are represented by their cumulative distribution functions $F(\cdot)$ (where $F(x) = \text{prob}(\tilde{x} \leqslant x)$), and the expected utility preference function takes the form $V(F) \equiv \int U(x)\,dF(x)$. (When $F(\cdot)$ possesses a density function $f(\cdot) \equiv F'(\cdot)$ this integral can be equivalently written as $\int U(x)f(x)\,dx$.) When the outcomes are multivariate commodity bundles of the form $(z_1, \ldots, z_n)$, $V(\cdot)$ takes the form $\int \ldots \int U(z_1, \ldots, z_n)\,dF(z_1, \ldots, z_n)$ over multivariate cumulative distribution functions $F(\cdot, \ldots, \cdot)$. The expected utility model derives its name from the fact that in each case, the preference function $V(\cdot)$ consists of the mathematical expectation of the von Neumann–Morgenstern utility function $U(\cdot)$, $U(\cdot, \ldots, \cdot)$, or utility index $\{U_i\}$ with respect to the probability distribution $F(\cdot)$, $F(\cdot, \ldots, \cdot)$, or P.

Mathematically, the hypothesis that the preference function $V(\cdot)$ takes the form of a statistical expectation is equivalent to the condition that it be 'linear in the probabilities'; that is, either a weighted sum of the components of P (i.e. $\Sigma U_i p_i$) or else a weighted integral of the functions $F(\cdot)$ or $f(\cdot)$[$\int U(x)\,dF(x)$ or $\int U(x)f(x)\,dx$]. Although this still allows for a wide variety of attitudes towards risk, depending upon the shape of the von Neumann–Morgenstern utility function $U(\cdot)$ or index $\{U_i\}$, the restriction that $V(\cdot)$ be linear in the probabilities is the primary empirical feature of the expected utility model and provides the basis for many of its observable implications and predictions.

It is important to distinguish between the preference function $V(\cdot)$ and the von Neumann–Morgenstern utility function $U(\cdot)$ (or index $\{U_i\}$) of an expected utility maximizer, in particular with regard to the prevalent though mistaken belief that expected utility preferences are somehow 'cardinal' in a sense which is not exhibited by preferences over non-stochastic commodity bundles. As with any real-valued representation of a preference ordering, an expected utility preference function $V(\cdot)$ is 'ordinal' in that it may be subject to any increasing transformation without affecting the validity of the representation; thus, for example, if $V(F) \equiv \int U(x)\,dF(x)$ represents the preferences of some expected utility maximizer, so will the (nonlinear) preference function $Y(F) \equiv [\int U(x)\,dF(x)]^3$. On the other hand, the von Neumann–Morgenstern utility functions which generate these preference functions are 'cardinal' in the sense that a function $U^*(\cdot)$ will generate an ordinally equivalent linear preference function $V^*(F) \equiv \int U^*(x)\,dF(x)$ if and only if it satisfies the cardinal relationship $U^*(x) \equiv a \cdot U(x) + b$ for some $a > 0$ (in which case $V^*(\cdot) = a \cdot V(\cdot) + b$). However, such situations also occur in the theory of preferences over non-stochastic commodity bundles: the Cobb–Douglas preference function $\alpha \cdot \ln(x) + \beta \cdot \ln(y) + \gamma \cdot \ln(z)$ (written here in its additive form) can be subject to any increasing transformation and is clearly ordinal, even though a vector of parameters $(\alpha^*, \beta^*, \gamma^*)$ will generate an ordinally equivalent additive form $\alpha^* \cdot \ln(x) + \beta^* \cdot \ln(y) + \gamma^* \cdot \ln(z)$ if and only if it satisfies the cardinal relationship $(\alpha^*, \beta^*, \gamma^*) = \lambda \cdot (\alpha, \beta, \gamma)$ for some $\lambda > 0$.

In the case of a simple outcome set of the form $\{x_1, x_2, x_3\}$, it is possible to illustrate the 'linearity in the probabilities' property of an expected utility maximizer's preferences over lotteries. Since every probability distribution (p_1, p_2, p_3) over this set must satisfy the condition $\Sigma p_i = 1$, we may represent each such distribution by a point in the unit triangle in the (p_1, p_3) plane, with p_2 given by $p_2 = 1 - p_1 - p_3$ (Figures 1 and 2). Since they represent the loci of solutions to the equations

$$U_1 p_1 + U_2 p_2 + U_3 p_3 = U_2 - [U_2 - U_1] \cdot p_1 + [U_3 - U_2] \cdot p_3$$
$$= \text{constant}$$

for the fixed utility indices $\{U_1, U_2, U_3\}$, the indifference curves of an expected utility maximizer consist of parallel straight lines in the triangle of slope $[U_2 - U_1]/[U_3 - U_2]$, as illustrated by the solid lines in Figure 1. An example of indifference curves which do *not* satisfy the expected utility hypothesis (i.e. are not linear in the probabilities) is given by the solid curves in Figure 2.

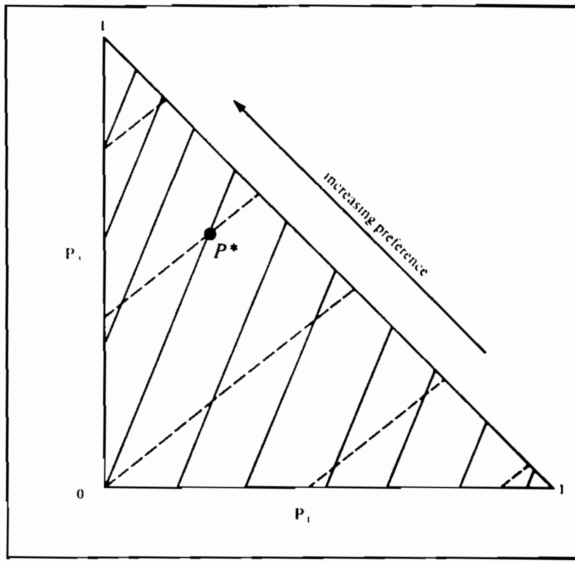

Figure 1 Expected Utility Indifference Curves

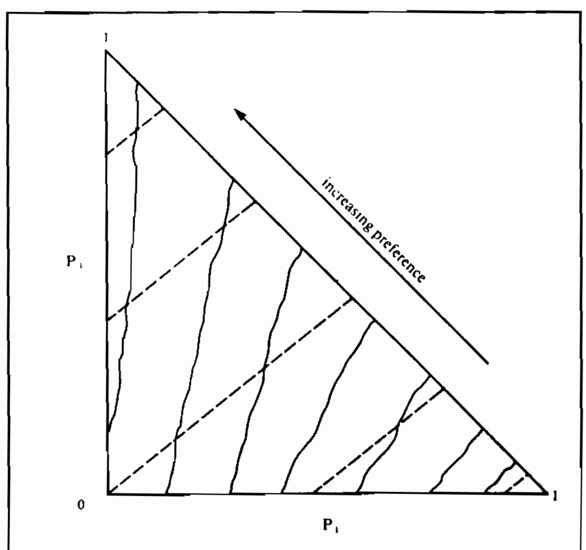

Figure 2 Non-Expected Utility Indifference Curves

When the outcomes $\{x_1, x_2, x_3\}$ represent different levels of wealth with $x_1 < x_2 < x_3$, this diagram can be used to illustrate other possible aspects of an expected utility maximizer's attitudes toward risk. On the general principle that more wealth is better, it is typically postulated that any change in a distribution (p_1, p_2, p_3) which increases p_3 at the expense of p_2, increases p_2 at the expense of p_1, or both, will be preferred by the individual: this property is known as 'first-order stochastic dominance preference'. Since such shifts of probability mass are represented by north, west or north-west movements in the diagram, first-order stochastic dominance preference is equivalent to the condition that indifference curves are upward sloping, with more preferred indifference curves lying to the north-west. Algebraically, this is equivalent to the condition $U_1 < U_2 < U_3$.

Another widely (though not universally hypothesized aspect of attitudes towards risk is that of 'risk aversion' (e.g. Arrow, 1974, ch. 3; Pratt, 1964). To illustrate this property of preferences, consider the dashed lines in Figure 1, which represent loci of solutions to the equations

$$x_1 p_1 + x_2 p_2 + x_3 p_3 = x_2 - [x_2 - x_1]\cdot p_1 + [x_3 - x_2]\cdot p_3$$
$$= \text{constant}$$

and hence may be termed 'iso-expected *value* loci'. Since north-east movements along any of these loci consist of increasing the tail probabilities p_1 and p_3 at the expense of middle probability p_2 in a manner which preserves the mean of the distribution, they correspond to what are termed 'mean preserving increases in risk' (e.g. Rothschild and Stiglitz, 1970, 1971). An individual is said to be 'risk averse' if such increases in risk always lead to less preferred indifference curves, which is equivalent to the graphical condition that the indifference curves be steeper than the iso-expected value loci. Since the slope of the latter is given by $[x_2 - x_1]/[x_3 - x_2]$, this is equivalent to the algebraic condition that $[U_2 - U_1]/[x_2 - x_1] > [U_3 - U_2]/[x_3 - x_2]$. Conversely, individuals who *prefer* mean preserving increases in risk are termed 'risk loving': such individuals' indifference curves will be flatter than the iso-expected value loci, and their utility indices will satisfy $[U_2 - U_1]/[x_2 - x_1] < [U_3 - U_2]/[x_3 - x_2]$.

Note finally that the indifference map in Figure 1 indicates that the lottery P^* is indifferent to the origin, which represents the degenerate lottery yielding x_2 with certainty. In such a case the amount x_2 is said to be the 'certainty equivalent' of the lottery P^*. The fact that the origin lies on a lower iso-expected value locus than P^* reflects a general property of risk averse preferences, namely that the certainty equivalent of any lottery will always be less than its mean. (For risk lovers, the opposite is always the case.)

When the outcomes are elements of the real line, it is possible to represent the above (as well as other) aspects of preferences in terms of the shape of the von Neumann–Morgenstern utility function $U(\cdot)$, as seen in Figures 3 and 4. In each figure, consider the lottery which assigns the probabilities 2/3: 1/3 to the outcome levels x' and x'', respectively. The expected value of this lottery (i.e. the value $\bar{x} = 2/3 \cdot x' + 1/3 \cdot x''$) is seen to lie between these two values, two-thirds of the way towards x'. The expected *utility* of this lottery – i.e. the value $\bar{u} = 2/3 \cdot U(x') + 1/3 \cdot U(x'')$ – is similarly seen to lie between $U(x')$ and $U(x'')$ on the vertical axis, two-thirds of the way towards $U(x')$. The point $(\bar{x}, \bar{u})$ will accordingly lie on the line segment connecting the points $(x', U(x'))$ and $(x'', U(x''))$, two-thirds of the way towards the former. In each figure, the certainty equivalent of this lottery is given by that sure outcome c which also yields a utility level of $\bar{u}$.

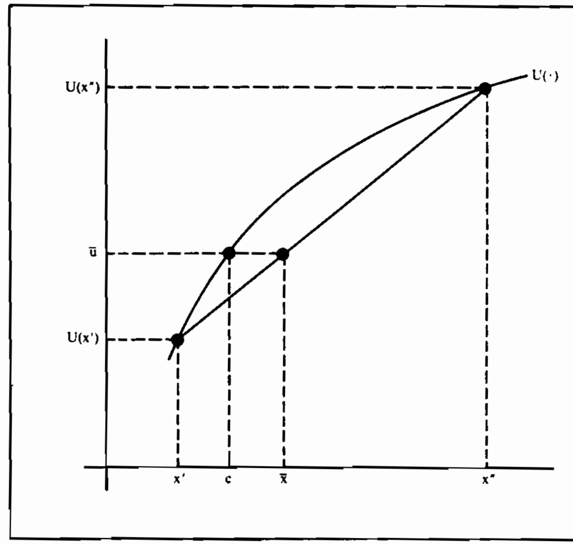

Figure 3 Von Neumann-Morgenstern Utility Function of a Risk Averse Individual

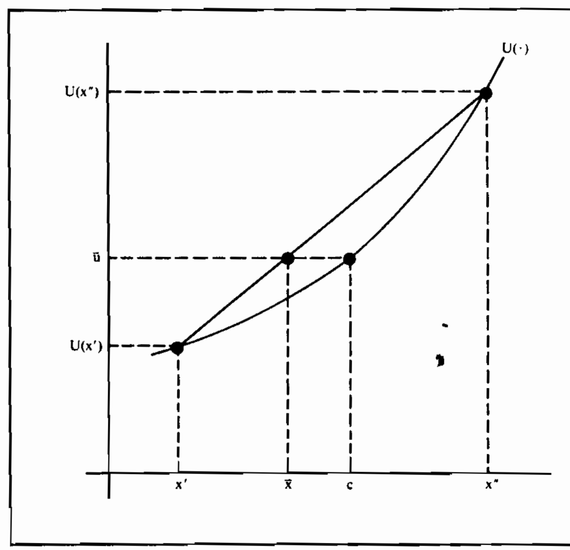

Figure 4 Von Neumann-Morgenstern Utility Function of a Risk Loving Individual

It is clear from our definition of first-order stochastic dominance preference above that this property of preferences can be extended to the case of density functions $f(\cdot)$ or cumulative distribution functions $F(\cdot)$ over the real line (e.g. Quirk and Saposnik, 1962), and that it is equivalent to the condition that $U(x)$ be an increasing function of x, as in Figures 3 and 4. It is also possible to generalize the notion of a mean preserving increase in risk to density functions or cumulative distribution functions (e.g. Rothschild and Stiglitz, 1970, 1971), and our earlier algebraic condition for risk aversion generalizes to the condition that $U''(x) < 0$ for all x, i.e. that the von Neumann–Morgenstern utility function $U(\cdot)$ be concave, as in Figure 3. As before, the property of risk aversion implies that the certainty equivalent c of any lottery will always lie below its mean, as seen in Figure 3, and once again, the opposite is true for the convex utility function of a risk lover, as seen in Figure 4. Two of the earliest and most important graphical analyses of risk attitudes in terms of the shape of the von Neumann–Morgenstern utility function are those of Friedman and Savage (1948) and Markowitz (1952).

The tremendous analytic capabilities of the expected utility model for the study of behaviour towards risk derive largely from the work of Arrow (1974) and Pratt (1964). Roughly speaking, these researchers showed that the 'degree' of concavity of the von Neumann–Morgenstern utility function can be used to provide a measure of an expected utility maximizer's 'degree' of risk aversion. Formally, the Arrow–Pratt characterization of comparative risk aversion is the result that the following conditions on a pair of (increasing, twice differentiable) von Neumann–Morgenstern utility functions $U_a(\cdot)$ and $U_b(\cdot)$ are equivalent:

$U_a(\cdot)$ is a concave transformation of $U_b(\cdot)$ (i.e.

$U_a(x) \equiv \rho[U_b(x)]$ for some increasing concave function $\rho(\cdot)$),
$-U_a''(x)/U_a'(x) \geqslant -U_b''(x)/U_b'(x)$, for each x, and

if c_a and c_b solve

$$U_a(c_a) = \int U_a(x)\,\mathrm{d}F(x) \quad \text{and} \quad U_b(c_b) = \int U_b(x)\,\mathrm{d}F(x)$$

for some distribution $F(\cdot)$, then $c_a \leqslant c_b$,

and if $U_a(\cdot)$ and $U_b(\cdot)$ are both concave, these conditions are in turn equivalent to:

if $r > 0$, $E[\tilde{z}] > r$, and α_a and α_b maximize

$$\int U_a[(I-\alpha)r + \alpha z]\,\mathrm{d}F(z) \quad \text{and} \quad \int U_b[(I-\alpha)r + \alpha z]\,\mathrm{d}F(z)$$

respectively, then $\alpha_a \leqslant \alpha_b$.

The first two of these conditions provide equivalent formulations of the notion that $U_a(\cdot)$ is a more concave function than $U_b(\cdot)$. In particular, the curvature measure $R(x) \equiv -U''(x)/U'(x)$ is known as the 'Arrow–Pratt index of (absolute) risk aversion', and plays a key role in the analytics of the expected utility model. The third condition states that the more risk averse utility function $U_a(\cdot)$ will never assign a higher certainty equivalent to any lottery $F(\cdot)$ than will $U_b(\cdot)$. The final condition pertains to the individuals' respective demands for risky assets. Specifically, assume that each of them must allocate $\$I$ between two assets, one yielding a riskless (gross) return of r per dollar, and the other yielding a risky return $\tilde{z}$ with a higher expected value. This condition thus says that the less risk averse utility function $U_b(\cdot)$ will generate at least as great a demand for the risky asset than the more risk averse utility function $U_a(\cdot)$. It is important to note that it is the *equivalence* of the above certainty equivalent and asset demand conditions which makes the Arrow–Pratt characterization such an important result in expected utility theory. (See Ross, 1981, however, for an alternative and stronger characterization of comparative risk aversion.)

Although the applications of the expected utility model extend to virtually all branches of economic theory (e.g. Hey, 1979), much of the flavour of these analyses can be gleaned from Arrow's (1974, ch. 3) analysis of the portfolio problem of the previous paragraph: rewriting $(I-\alpha)r + \alpha z$ as $Ir + \alpha\cdot(z-r)$, the first-order condition for this problem can be expressed as:

$$\int z\cdot U'[Ir + \alpha\cdot(z-r)]\,\mathrm{d}F(z) - r\cdot\int U'[Ir + \alpha\cdot(z-r)]\,\mathrm{d}F(z) = 0,$$

that is, the marginal *expected* utility of the last dollar allocated to each asset is the same. The second-order condition can be written as:

$$\int (z-r)^2\cdot U''[Ir + \alpha\cdot(z-r)]\,\mathrm{d}F(z) < 0$$

and is ensured by the property of risk aversion [i.e. $U''(\cdot) < 0$].

As usual, we may differentiate the first-order condition to obtain the effect of a change in some parameter, say initial wealth I, on the optimal level of investment in the risky asset (i.e. on the optimal value of α). Differentiating the first-order condition (including α) with respect to I, solving for $\mathrm{d}\alpha/\mathrm{d}I$, and invoking the second-order condition and the positivity of r yields that this effect possesses the same sign as:

$$\int (z-r)\cdot U''[Ir + \alpha\cdot(z-r)]\,\mathrm{d}F(z).$$

Making the substitution $U''(\cdot) \equiv -R(\cdot)\cdot U'(\cdot)$ and subtracting $R(Ir)$ times the first-order condition yields that this term is equal to:

$$-\int (z-r)\cdot\{R[Ir + \alpha\cdot(z-r)] - R(I)\} \times U'[Ir + \alpha\cdot(z-r)]\,\mathrm{d}F(z).$$

On the assumption that α is positive and $R(\cdot)$ is monotonic, the expression $(z-r)\cdot[R(Ir + \alpha\cdot(z-r)) - R(Ir)]$ will possess the same sign as $R'(\cdot)$. This implies that the derivative $\mathrm{d}\alpha/\mathrm{d}I$ will always be positive (negative) whenever the Arrow–Pratt index $R(x)$ is a decreasing (increasing) function of the individual's wealth level x. In other words, an increase in initial wealth will always increase (decrease) the demand for the risky asset if and only if $U(\cdot)$ exhibits decreasing (increasing) absolute risk aversion in wealth. Further examples of the analytics of risk and risk aversion in the expected utility model may be found in the above references as well as the surveys of Hirshleifer and Riley (1979), Lippman and McCall (1981) and Machina (1983b).

Finally, in addition to the case of preferences over probability distributions, it is also possible to refer to expected utility preferences over alternative 'state-payoff bundles' (e.g. Hirshleifer, 1965, 1966). This approach postulates a (typically finite) set of 'states of nature' (i.e. a mutually exclusive and exhaustive partition of the set of observable occurrences) and the objects of choice consist of state-payoff bundles of the form $(x_1, \ldots, x_n)$, where x_i denotes the outcome the individual will receive should state i occur. An expected utility maximizer whose subjective probabilities of the n states are given by the values $(\bar{p}_1, \ldots, \bar{p}_n)$ will rank such bundles according to the preference function $V(x_1, \ldots, x_n) \equiv \Sigma U(x_i)\bar{p}_i$, or in the event that the utility of wealth function $U_i(\cdot)$ itself depends upon the state of nature, according to the 'state-dependent' preference function $V(x_1, \ldots, x_n) \equiv \Sigma U_i(x_i)\bar{p}_i$ (e.g. Karni, 1985). One of the advantages of the general 'state-preference' approach is that it does not require that we be able to observe the individual's probabilistic beliefs, or that different individuals share the same probabilistic beliefs.

AXIOMATIC DEVELOPMENT. Although there exist dozens of formal axiomatizations of the expected utility model in its different contexts, most proceed by specifying an outcome space and postulating that the individual's preferences over probability distributions on this outcome space satisfy the following four axioms: completeness, transitivity, continuity and the independence axiom. Although it is beyond the scope of this entry to provide a rigorous derivation of the expected utility model in its most general setting, it is possible to illustrate the meaning of the axioms and sketch a proof of the expected utility representation theorem in the simple case of a finite outcome set of the form $\{x_1, \ldots, x_n\}$.

Recall that in such a case the objects of choice consist of all probability distributions $P = (p_1, \ldots, p_n)$ over $\{x_1, \ldots, x_n\}$, so that the following axioms refer to the individuals' weak preference relation $\succsim$ over this set, where $P^* \succsim P$ is read 'P^* is weakly preferred (i.e. preferred or indifferent) to P' (the associated strict preference relation $\succ$ and indifference relation $\sim$ are defined in the usual manner):

Completeness: For any two distributions P and P^* either $P^* \succsim P$, $P \succsim P^*$, or both.

Transitivity: If $P^{**} \succsim P^*$ and $P^* \succsim P$, then $P^{**} \succsim P$,

Mixture Continuity: If $P^{**} \succsim P^* \succsim P$, then there exists some $\lambda \in [0, 1]$ such that $P^* \sim \lambda P^{**} + (1-\lambda)P$, and

Independence: For any two distributions P and P^*, $P^* \succsim P$ if and only if $\lambda P^* + (1-\lambda)P^{**} \succsim \lambda P + (1-\lambda)P^{**}$ for all $\lambda \in (0, 1]$ and all P^{**},

where $\lambda P + (1-\lambda)P^*$ denotes the 'probability mixture' of P and P^*, i.e., the lottery with probabilities

$$(\lambda p_1 + (1-\lambda)p_1^*, \ldots, \lambda p_n + (1-\lambda)p_n^*).$$

The notion of a probability mixture is closely related (though not identical) to that of a 'compound lottery', in the sense that the probability mixture $\lambda P + (1-\lambda)P^*$ yields the same probabilities of ultimately obtaining the outcomes $\{x_1, \ldots, x_n\}$ as would a compound lottery yielding a $\lambda:(1-\lambda)$ chance of obtaining the respective lotteries P or P^*.

The completeness and transitivity axioms are completely analogous to their counterparts in the standard theory of the consumer (in particular, transitivity of $\succsim$ can be shown to imply transitivity of both $\succ$ and $\sim$). Mixture continuity states that if the lottery P^{**} is weakly preferred to P^*, and P^* is weakly preferred to P, then there will exist some probability mixture of the most and least preferred lotteries which is indifferent to the intermediate one.

As in standard consumer theory, completeness, transitivity and continuity serve essentially to establish the existence of a real-valued preference function $V(p_1, \ldots, p_n)$, which represents the relation $\succsim$, in the sense that $P^* \succsim P$ if and only if $V(p_1^*, \ldots, p_n^*) \geqslant V(p_1, \ldots, p_n)$. It is the independence axiom which, besides forming the basis of its widespread normative appeal, gives the theory its primary empirical content by implying that the preference function must take the linear form $V(p_1, \ldots, p_n) \equiv \Sigma U_i p_i$. To see the meaning of this axiom, assume that one is always indifferent between a compound lottery and its probabilistically equivalent single-stage lottery, and that P^* happens to be weakly preferred to P. In that case, the choice between the mixtures $\lambda P^* + (1-\lambda)P^{**}$ and $\lambda P^* + (1-\lambda)P^{**}$ is equivalent to being presented with a coin that has a $(1-\lambda)$ chance of landing tails (in which case the prize will be P^{**}) and being asked *before the flip* whether one would rather win P or P^* in the event of a head. The normative argument for the independence axiom is that either the coin will land tails, in which case the choice would not have mattered, or it will land heads, in which case one is 'in effect' back to a choice between P and P^* and one 'ought' to have the same preferences as before. Note finally that the above statement of the axiom in terms of the weak preference relation $\succsim$ also implies its counterparts in terms of strict preference and indifference.

In the following sketch of the expected utility representation theorem, expressions such as '$x_i \succsim x_j$' should be read as saying that the individual weakly prefers the degenerate lottery yielding x_i with certainty to that yielding x_j with certainty, and '$\lambda x_i + (1-\lambda)x_j$' will be used to denote the $\lambda:(1-\lambda)$ probability mixture between these two degenerate lotteries, and so on.

The first step in the argument is to define the von Neumann–Morgenstern utility index $\{U_i\}$ and the expected utility preference function $V(\cdot)$. Without loss of generality, we may reorder the outcomes so that $x_n \succsim x_{n-1} \succsim \cdots \succsim x_2 \succsim x_1$. Since $x_n \succsim x_i \succsim x_1$ for each outcome x_i, we have by mixture continuity that there will exist scalars $\{U_i\} \subset [0, 1]$ such that $x_i \sim U_i x_n + (1-U_i)x_1$ for each i (note that we can define $U_1 = 0$ and $U_n = 1$). Given this, define $V(P)$ to equal $\Sigma U_i p_i$ for all P.

The second step is to show that each lottery $P = (p_1, \ldots, p_n)$ is indifferent to the mixture $\lambda x_n + (1-\lambda)x_1$ where $\lambda = \Sigma U_i p_i$. Since $(p_1, \ldots, p_n)$ can be written as the n-fold probability mixture $p_1 \cdot x_1 + p_2 \cdot x_2 + \cdots + p_n \cdot x_n$ and each outcome x_i is indifferent to the mixture $U_i x_n + (1-U_i)x_1$, an n-fold application of the independence axiom yields that $(p_1, \ldots, p_n)$ is indifferent to the mixture

$$p_1 \cdot [U_1 x_n + (1-U_1)x_1] + p_2 \cdot [U_2 x_n + (1-U_2)x_1] + \cdots$$
$$\cdots + p_n \cdot [U_n x_n + (1-U_n)x_1],$$

which is equal to $(\Sigma U_i p_i) \cdot x_n + (1 - \Sigma U_i p_i) \cdot x_1$.

The third step is to demonstrate that the mixture $\lambda x_n + (1-\lambda)x_1$ is weakly preferred to the mixture $\gamma x_n + (1-\gamma)x_1$ if and only if $\lambda \geqslant \gamma$. This follows immediately from the independence axiom and the fact that $\lambda \geqslant \gamma$ implies that these two lotteries may be expressed as the respective mixtures

$$(\lambda - \gamma) \cdot x_n + (1 - \lambda + \gamma) \cdot Q$$

and

$$(\lambda - \gamma) \cdot x_1 + (1 - \lambda + \gamma) \cdot Q,$$

where Q is defined as the mixture

$$[\gamma/(1-\lambda+\gamma)] \cdot x_n + [(1-\lambda)/(1-\lambda+\gamma)] \cdot x_1.$$

The completion of the proof is now simple. For any two distributions P^* and P, we have by transitivity and the second step that $P^* \succsim P$ if and only if

$$(\Sigma U_i p_i^*) \cdot x_n + (1 - \Sigma U_i p_i^*) \cdot x_1 \succsim (\Sigma U_i p_i) \cdot x_n + (1 - \Sigma U_i p_i) \cdot x_1,$$

which by the third step is equivalent to the condition that $\Sigma U_i p_i^* \geqslant \Sigma U_i p_i$, or in other words, that $V(P^*) \geqslant V(P)$.

As mentioned, the expected utility model has been axiomatized many times and in many contexts. The most comprehensive account of the axiomatics of the model is undoubtedly Fishburn (1982).

HISTORY. The hypothesis that individuals might maximize the expectation of 'utility' rather than of monetary value was first proposed independently by the mathematicians Gabriel Cramer and Daniel Bernoulli, in each case as the solution to a problem posed by Daniel's cousin Nicholas Bernoulli (see Bernoulli, 1738). This problem, which has since come to be known as the 'St Petersburg Paradox', considers the gamble which offers a 1/2 chance of \$1.00, a 1/4 chance of \$2.00, a 1/8 chance of \$4.00, and so on. Although the expected value of this prospect is

$$(1/2) \cdot \$1.00 + (1/4) \cdot (\$2.00) + (1/8) \cdot (\$4.00) + \cdots$$
$$\cdots = \$0.50 + \$0.50 + \$0.50 + \cdots = \$\infty,$$

common sense suggests that no one would be willing to forgo a very substantial certain payment in order to play it. Cramer and Bernoulli proposed that instead of looking at expected value, individuals might evaluate this and other lotteries by their 'expected utility', with utility given by a function such as the natural logarithm or the square root of wealth, in which case the certainty equivalent of the St Petersburg gamble becomes a moderate (and plausible) amount.

Two hundred years later, the St Petersburg Paradox was generalized by Karl Menger (1934), who noted that whenever the utility of wealth function was unbounded (as with the natural logarithm or square root functions), it would be possible to construct similar examples with infinite expected utility and hence infinite certainty equivalents (replace the payoffs \$1.00, \$2.00, \$4.00, ... in the above example by x_1, x_2, $x_3, \ldots$ where $U(x_i) = 2^i$ for each i). In light of this, von Neumann–Morgenstern utility functions are typically (though not universally) postulated to be bounded functions of wealth.

The earliest formal axiomatic treatment of the expected utility hypothesis was developed by Frank Ramsey (1926) as part of his theory of subjective probability or individuals' 'degrees of belief' in the truth of various alternative propositions. Starting from the premise that there exists an 'ethically neutral' proposition whose degree of belief is 1/2 and whose validity or invalidity is of no independent value, Ramsey proposed a set of axioms on how the individual would be willing to stake prizes on its truth or falsity in a manner which allowed for the derivation of the 'utilities' of these prizes. He then used these utility values and betting preferences to determine the individual's degrees of belief in other propositions. Perhaps because it was intended as a contribution to the philosophy of belief rather than the theory of risk bearing, Ramsey's analysis did not have the impact upon the economics literature that it deserved.

The first axiomatization of the expected utility model to receive widespread attention was that of John von Neumann and Oskar Morgenstern, which was presented in connection with their formulation of the theory of games (von Neumann and Morgenstern, 1944, 1947, 1953). Although both these developments were recognized as breakthroughs, the mistaken belief that von Neumann and Morgenstern had somehow mathematically overthrown the Hicks–Allen 'ordinal revolution' led to some confusion until the difference between 'utility' in the von Neumann–Morgenstern and ordinal (i.e. non-stochastic) senses was illuminated by writers such as Ellsberg (1954) and Baumol (1958).

Another factor which delayed the acceptance of the theory was the lack of recognition of the role played by the independence axiom, which did not explicitly appear in the von Neumann–Morgenstern formulation. In fact, the initial reaction of researchers such as Baumol (1951) and Samuelson (1950) was that there was no reason why preferences over probability distributions must *necessarily* be linear in the probabilities. However the independent discovery of the independence axiom by Marschak (1950), Samuelson (1952) and others, and Malinvaud's (1952) observation that it had been implicitly invoked by von Neumann and Morgenstern, led to an almost universal acceptance of the expected utility hypothesis as both a normative and positive theory of behaviour toward risk. Practically the only dissenting voice was that of Maurice Allais, whose famous paradox (see below) and other empirical and theoretical work (e.g. Allais, 1952) has provided the basis for the resurgence of interest in alternatives to expected utility in the late 1970s and 1980s. This period also saw the development of the elegant axiomatization of Herstein and Milnor (1953) as well as Savage's (1954) joint axiomatization of utility and subjective probability, which formed the basis of the state-preference approach described above.

While the 1950s essentially saw the completion of foundational work on the expected utility model, the 1960s and 1970s saw the flowering of its analytic capabilities and its application to fields such as portfolio selection (Merton, 1969), optimal savings (Levhari and Srinivasan, 1969), international trade (Batra, 1975), and even the measurement of inequality (Atkinson, 1970). This movement was spearheaded by the development of the Arrow–Pratt characterization of risk aversion (see above) and the characterization, by Rothschild–Stiglitz (1970, 1971) and others, of the notion of 'increasing risk'. This latter work in turn led to the development of a general theory of 'stochastic dominance' (e.g. Whitmore and Findlay, 1978), which further expanded the analytical powers of the model.

Although the expected utility model received a small amount of experimental testing by economists in the early 1950s (e.g. Mosteller and Nogee, 1951; Allais, 1952) and continued to be examined by psychologists, interest in the empirical validity of the model waned from the mid-1950s through the mid-1970s, no doubt due to both the normative appeal of the independence axiom and model's analytical successes. However, the late 1970s and 1980s have witnessed a revival of interest in the testing of the expected utility model; a growing body of evidence that individuals' preferences *systematically* depart from linearity in the probabilities; and the development, analysis and application of alternative models of choice under risk (see below). It is fair to say that today the debate over the descriptive (and even normative) validity of the expected utility hypothesis is more extensive than it has been in 30 years, and the outcome of this debate will have important implications for the direction of research in the economic theory of individual behaviour towards risk.

EVIDENCE AND ALTERNATIVE HYPOTHESES. As mentioned above, the current body of experimental evidence suggests that individual preferences over lotteries are typically *not* linear in the probabilities, but rather depart systematically from this property. The earliest, and undoubtedly best-known, example of this is the so-called 'Allais paradox' (Allais, 1952), in which the individual is asked to rank each of the following pairs of prospects (where \$1M = \$1,000,000):

$$a_1: \{1.00 \text{ chance of \$1M} \quad \text{versus} \quad a_2: \begin{cases} 0.10 \text{ chance of \$5M} \\ 0.89 \text{ chance of \$1M} \\ 0.01 \text{ chance of \$0,} \end{cases}$$

and:

$$a_3: \begin{cases} 0.10 \text{ chance of \$5M} \\ 0.90 \text{ chance of \$0} \end{cases} \quad \text{versus} \quad a_4: \begin{cases} 0.11 \text{ chance of \$1M} \\ 0.89 \text{ chance of \$0.} \end{cases}$$

Since each of these lotteries involves outcomes in the set $\{x_1, x_2, x_3\} = \{\$0, \$1M, \$5M\}$, they may be plotted in the (p_1,p_3) triangle diagram, as illustrated in Figures 5 and 6. The fact that the four prospects form a parallelogram in this triangle makes this problem a useful test of linearity (i.e. the expected utility hypothesis), since it implies that an expected utility maximizer will prefer a_1 to a_2 if and only if he or she prefers a_4 to a_3 (algebraically, this is in turn equivalent to the inequality $(0.10 \cdot U(\$5M) - 0.11 \cdot U(\$1M) + 0.01 \cdot U\$(0) < 0)$.

However, experimenters such as Allais (1952), Morrison (1967), Moskowitz (1974), Raiffa (1968), Slovic and Tversky (1974) and others, have found that the modal if not majority choice was for a_1 in the first pair and a_3 in the second pair, as would be chosen by an individual whose indifference curves 'fanned out' as in Figure 6. Subsequent studies by Hagen (1979), Karmarkar (1974), MacCrimmon and Larsson (1979), McCord and de Neufville (1983) and others, using both similar and qualitatively different types of examples, have also revealed systematic departures from linearity in the direction of 'fanning out' (see Machina, 1983a, 1983b).

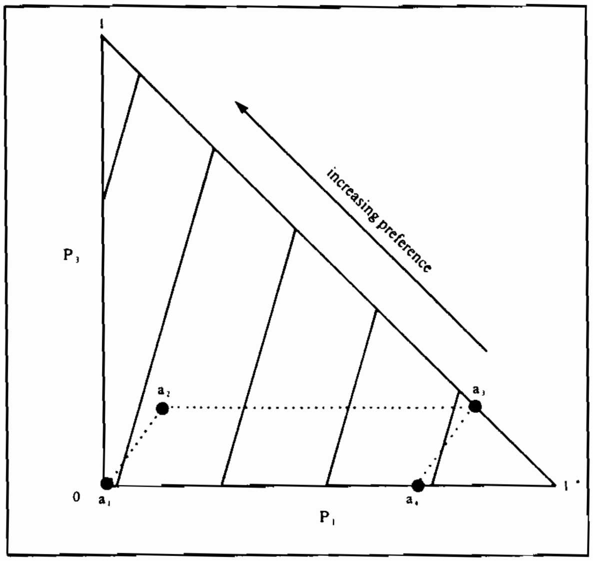

Figure 5 Allais Paradox with Expected Utility Indifference Curves

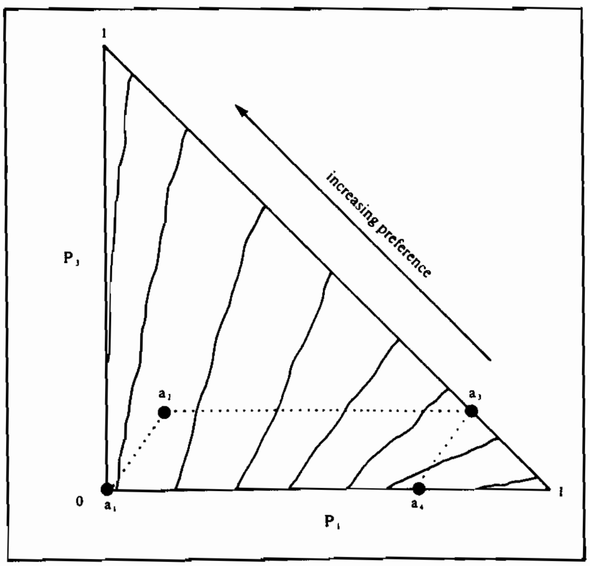

Figure 6 Allais Paradox with Non-Expected Utility Indifference Curves that "Fan Out"

In light of this evidence, researchers have begun to develop alternatives to the expected utility model (typically generalizations of it) which are capable of exhibiting this form of nonlinearity as well as other standard properties of risk preferences such as first-order stochastic dominance preference and risk aversion. (A set of non-expected utility indifference curves which exhibits these three properties, for example, is given in Figure 2.) Specific nonlinear functional forms for preference functions which have been proposed include those of Edwards (1955) and Kahneman and · Tversky (1979) ($\Sigma U(x_i)\pi(p_i)$); Chew and MacCrimmon (1979) and Chew (1983) $\{[\int U(x)\,dF(x)]/[\int W(x)\,dF(x)]\}$; and Quiggin (1982) $\{\int U(x)\,dG[F(x)]\}$. A general framework for the analysis of differentiable non-expected utility preference functions in terms of their local linear approximations, which can be interpreted as local 'expected utility' approximations, is developed in Machina (1982, 1983a). Finally, the findings by Lichtenstein and Slovic (1971), Grether and Plott (1979) and others of systematic intransitivities in preferences over lotteries (but see Karni and Safra, 1984), have led to the development of non-transitive models by researchers such as Bell (1982), Fishburn (1983), and Loomes and Sugden (1982). (For a more complete survey of the experimental evidence on the expected utility hypothesis as well as alternative models of behaviour towards risk, see Machina, 1983b.)

MARK J. MACHINA

See also BERNOULLI, DANIEL; DECISION THEORY; RAMSEY, FRANK PLUMPTON; REPRESENTATION OF PREFERENCES; RISK UNCERTAINTY; UTILITY THEORY AND DECISION-MAKING.

BIBLIOGRAPHY

Allais, M. 1952. Fondements d'une théorie positive des choix comportant un risque et critique des postulats et axiomes de l'école Américaine. *Colloques Internationaux du Centre National de la Recherche Scientifique* 40, (1953), 257–332. Trans. as: The foundations of a positive theory of choice involving risk and a criticism of the postulates and axioms of the American School, in Allais and Hagen (1979).

Allais, M. and Hagen, O. (eds) 1979. *Expected Utility Hypotheses and the Allais Paradox*. Dordrecht: D. Reidel.

Arrow, K. 1974. *Essays in the Theory of Risk-Bearing*. Amsterdam: North-Holland.

Atkinson, A. 1970. On the measurement of inequality. *Journal of Economic Theory* 2(3), September, 244–63.

Batra, R. 1975. *The Pure Theory of International Trade under Uncertainty*. London: Macmillan.

Baumol, W. 1951. The Neumann–Morgenstern utility index: an ordinalist view. *Journal of Political Economy* 59(1), February, 61–6.

Baumol, W. 1958. The cardinal utility which is ordinal. *Economic Journal* 68, December, 665–72.

Bell, D. 1982. Regret in decision making under uncertainty. *Operations Research* 30, September–October, 961–81.

Bernoulli, D. 1738. Specimen theoriae novae de mensura sortis. *Commentarii Academiae Scientiarum Imperialis Petropolitanae*. Trans. as: Exposition of a new theory on the measurement of risk, *Econometrica* 22, January 1954, 23–36.

Chew, S.H. 1983. A generalization of the quasilinear mean with applications to the measurement of income inequality and decision theory resolving the Allais paradox. *Econometrica* 51(4), July, 1065–92.

Chew, S. and MacCrimmon, K. 1979. Alpha–Nu choice theory: a generalization of expected utility theory. University of British Columbia Faculty of Commerce and Business Administration Working Paper No. 669, July.

Debreu, G. 1959. *Theory of Value: An Axiomatic Analysis of Economic Equilibrium*. New Haven: Yale University Press.

Edwards, W. 1955. The prediction of decisions among bets. *Journal of Experimental Psychology* 50(3), September, 201–14.

Ellsberg, D. 1954. Classical and current notions of 'measurable utility'. *Economic Journal* 64, September, 528–56.

Fishburn, P. 1982. *The Foundations of Expected Utility*. Dordrecht: D. Reidel.

Fishburn, P. 1983. Nontransitive measurable utility. *Journal of Mathematical Psychology* 26(1), August, 31–67.

Friedman, M. and Savage, L. 1948. The utility analysis of choices involving risk. *Journal of Political Economy* 56, August, 279–304. Reprinted in *Readings in Price Theory*, ed. G. Stigler and K. Boulding, London: George Allen & Unwin, 1953.
Grether, D. and Plott, C. 1979. Economic theory of choice and the preference reversal phenomenon. *American Economic Review* 69(4), September, 623–38.
Hagen, O. 1979. Towards a positive theory of preferences under risk. In Allais and Hagen (1979).
Herstein, I. and Milnor, J. 1953. An axiomatic approach to measurable utility. *Econometrica* 21, April, 291–7.
Hey, J. 1979. *Uncertainty in Microeconomics*. Oxford: Martin Robinson; New York: New York University Press.
Hirshleifer, J. 1965. Investment decision under uncertainty: choice theoretic approaches. *Quarterly Journal of Economics* 79, November, 509–36.
Hirshleifer, J. 1966. Investment decision under uncertainty: applications of the state-preference approach. *Quarterly Journal of Economics* 80, May, 252–77.
Hirshleifer, J. and Riley, J. 1979. The analytics of uncertainty and information – an expository survey. *Journal of Economic Literature* 17(4), December. 1375–421.
Kahneman, D. and Tversky, A. 1979. Prospect theory: an analysis of decision under risk. *Econometrica* 47(2), March, 263–91.
Karmarkar, U. 1974. The effect of probabilities on the subjective evaluation of lotteries. Massachusetts Institute of Technology Sloan School of Management Working Paper No. 698–74, February.
Karni, E. 1985. *Decision Making under Uncertainty: the Case of State-Dependent Preferences*. Cambridge, Mass.: Harvard University Press.
Karni, E. 1985. Increasing risk with state dependent preferences. *Journal of Economic Theory* 35(1), 172–7.
Karni, E. and Safra, Z. 1984. 'Preference reversal' and the theory of choice under risk. Johns Hopkins University Working Papers in Economics No. 141.
Levhari, D. and Srinivasan, T.N. 1969. Optimal savings under uncertainty. *Review of Economic Studies* 36–2, April, 153–64.
Lichtenstein, S. and Slovic, P. 1971. Reversals of preferences between bids and choices in gambling decisions. *Journal of Experimental Psychology* 89(1), July, 46–55.
Lippman, S. and McCall, J. 1981. The economics of uncertainty: selected topics and probabilistic methods. In *Handbook of Mathematical Economics*, ed. K. Arrow and M. Intriligator, Vol. 1, Amsterdam: North-Holland.
Loomes, G. and Sugden, R. 1982. Regret theory: an alternative theory of rational choice under uncertainty. *Economic Journal* 92. (368), December, 805–24.
McCord, M. and de Neufville, R. 1983. Empirical demonstration that expected utility analysis is not operational. In Stigum and Wenstøp (1983).
MacCrimmon, K. and Larsson, S. 1979. Utility theory: axioms versus 'paradoxes'. In Allais and Hagen (1979).
Machina, M. 1982. 'Expected utility' analysis without the independence axiom. *Econometrica* 50(2), March, 277–323.
Machina, M. 1983a. Generalized expected utility analysis and the nature of observed violations of the independence axiom. In Stigum and Wenstøp (1983).
Machina, M. 1983b. The economic theory of individual behavior toward risk: theory, evidence and new directions. Institute for Mathematical Studies in the Social Sciences Technical Report No. 433, Stanford University, October.
Malinvaud, E. 1952. Note on von Neumann–Morgenstern's strong independence axiom. *Econometrica* 20(4), October, 679.
Markowitz, H. 1952. The utility of wealth. *Journal of Political Economy* 60, April, 151–8.
Marschak, J. 1950. Rational behavior, uncertain prospects, and measurable utility. *Econometrica* 18, April, 111–41 (Errata, July 1950).
Menger, K. 1934. Das Unsicherheitsmoment in der Wertlehre. *Zeitschrift für Nationalökonomie*. Trans. as: The role of uncertainty in economics, in *Essays in Mathematical Economics in Honor of Oskar Morgenstern*, ed. M. Shubik, Princeton: Princeton University Press, 1967.
Merton, R. 1969. Lifetime portfolio selection under uncertainty: the continuous time case. *Review of Economics and Statistics* 51(3), August, 247–57.
Morrison, D. 1967. On the consistency of preferences in Allais' paradox. *Behavioral Science* 12(5), September, 373–83.
Moskowitz, H. 1974. Effects of problem representation and feedback on rational behavior in Allais and Morlat-type problems. *Decision Sciences* 2.
Mosteller, F. and Nogee, P. 1951. An experimental measurement of utility. *Journal of Political Economy* 59, October, 371–404.
Pratt, J. 1964. Risk aversion in the small and in the large. *Econometrica* 32, January–April, 122–36.
Quiggin, J. 1982. A theory of anticipated utility. *Journal of Economic Behavior and Organization* 3(4), December, 323–43.
Quirk, J. and Saposnick, R. 1962. Admissibility and measurable utility functions. *Review of Economic Studies* 29, February, 140–46.
Raiffa, H. 1968. *Decision Analysis: Introductory Lectures on Choice under Uncertainty*. Reading, Mass.: Addison Wesley.
Ramsey, F. 1926. Truth and probability. In *The Foundations of Mathematics and Other Logical Essays*, ed. R. Braithwaite, New York: Harcourt, Brace and Co., 1931. Reprinted in *Foundations: Essays in Philosophy, Logic, Mathematics and Economics*, ed. D. Mellor, New Jersey: Humanities Press, 1978.
Ross, S. 1981. Some stronger measures of risk aversion in the small and in the large, with applications. *Econometrica* 49(3), May, 621–38.
Rothschild, M. and Stiglitz, J. 1970. Increasing risk I: a definition. *Journal of Economic Theory* 2(3), September, 225–43.
Rothschild, M. and Stiglitz, J. 1971. Increasing risk II: its economic consequences. *Journal of Economic Theory* 3(1), March, 66–84.
Safra, Z. 1985. Existence of equilibrium for Walrasian endowment games. *Journal of Economic Theory* 37(2), 366–78.
Samuelson, P. 1950. Probability and attempts to measure utility. *Economic Review* 1, July, 167–73. Reprinted in Stiglitz (1965).
Samuelson, P. 1952. Probability, utility, and the independence axiom. *Econometrica* 20, October, 670–78. Reprinted in Stiglitz (1965).
Savage, L. 1954. *The Foundations of Statistics*. New York: John Wiley & Sons. Enlarged and revised edn, New York: Dover, 1972.
Slovic, P. and Tversky, A. 1974. Who accepts Savage's Axiom? *Behavioral Science* 19(6), November, 368–73.
Stiglitz, J. (ed.) 1965. *Collected Scientific Papers of Paul A. Samuelson*, Vol. 1. Cambridge, Mass.: MIT Press.
Stigum, B. and Wenstøp, F. (eds) 1983. *Foundations of Utility and Risk Theory with Applications*. Dordrecht: D. Reidel.
von Neumann, J. and Morgenstern, O. 1944. *Theory of Games and Economic Behavior*. Princeton: Princeton University Press. 2nd edn, 1947; 3rd edn, 1953.
Whitmore, G. and Findlay, M. (eds) 1978. *Stochastic Dominance: An Approach to Decision Making Under Risk*. Lexington, Mass.: D.C. Heath.

expenditure functions. *See* COST FUNCTIONS; DUALITY.

expenditure tax. The idea of an expenditure tax has a long ancestry, dating back at least to Hobbes, who argued that people should be taxed according to the resources of the community they absorb not according to what they contribute. The case was later taken up by J.S. Mill, Marshall, Pigou and Irving Fisher. In modern times, the advocacy of an expenditure tax is most associated with the Cambridge economist Nicholas Kaldor (1955). Recently it has been espoused by the Meade Committee (Meade, 1978), and separately by two members of that Committee, Kay and King (1978).

There are efficiency and equity arguments for considering an expenditure tax as an alternative to income taxation. As far as efficiency is concerned, there is a commonly held view that because income tax involves the double taxation of saving, and therefore lowers the rate of return on saving below the rate of