Practice Problems: First-Year M. Phil Microeconomics, Choice under Uncertainty
Vincent P. Crawford, University of Oxford **Michaelmas Term 2010**

Problems from Mas-Colell, Whinston, and Green, *Microeconomic Theory*, Oxford, 1995, chapter 6 (note that some exercises are in the text within the chapter):

Exercises 6.B.1-2,4,7
Exercises 6.C.1-8, 10-13, 16-18, 20
Exercises 6.D.1-4
Exercises 6.E.2-3.

Additional problems of my own (from various sources):

1. Give both an algebraic demonstration, and a graphical depiction in the probability triangle (as in MWG's Figure 6.B.1), of why the following set of lottery preferences violate the expected utility hypothesis. In your graph, stick with MWG's axis labels and the convention $x_3 > x_2 > x_1$ for final money wealth levels ("P" means "strictly prefers"):

[3000 with probability 0.90, 0 with probability 0.10] P [6000 with probability 0.45, 0 with probability 0.55] but

[6000 with probability 0.001, 0 with probability 0.999] P [3000 with probability 0.002, 0 with probability 0.998]

2. Suppose that an individual must choose among distributions some of which are unbounded (the sets of outcomes to which they assign positive probability are unbounded), and that the vN-M utilities that rationalize his preferences over bounded probability distributions make the expected utility of some of these unbounded distributions infinite. Assume that his preferences are consistent with expected-utility maximization, in the sense that it determines his preferences in the usual way when the alternatives in question have finite expected utilities; and that he always prefers a distribution with infinite expected utility to one with finite expected utility.

(a) Show that his preferences over probability distributions cannot be continuous in the sense of the axiom used in the vN-M theorem. (Hint: suppose that the distribution P has infinite expected utility, but that 0-for-certain and 5-for-certain have finite expected utilities. Then consider the individual's preferences between the alternatives 5-for-certain and [aP + (1-a)0] as a approaches zero.)

3. Suppose that there are two states of the world, s_1 and s_2, and that an individual who knows the probabilities, p_1 and p_2 respectively, of the two states chooses among state-contingent consumption bundles as if to maximize the expectation of a state-independent, strictly increasing vN-M utility function.

(a) Suppose that the individual is risk-neutral, and that he is indifferent between (8, 2) and (4, 4). What must the value of p_1 be?

(b) Now suppose that the individual may be either risk-averse or risk-loving. What is the lowest possible value of p_1 for which the individual could weakly (or strictly) prefer the state-contingent consumption bundle (6, 2) to the bundle (2, 6)? Do his risk preferences affect your answer? Why or why not?

(c) Now suppose that the individual is risk-averse, and that he is indifferent between (6, 2) and (2, 6). Show (graphically or algebraically, whichever you prefer) that he must weekly prefer (4, 4) to either of these bundles.

4. Does the vN-M utility function $U(x) = \ln(x) + x$ exhibit increasing, decreasing, or constant relative risk aversion?

5. Consider a risk-averse, expected-utility maximizing agent with vN-M utility function $u(\cdot)$ and initial wealth w.

(a) Show how to determine (by giving an expression that implicitly defines it) the minimum probability of winning, p, needed to get the agent to accept a binary bet in which the outcomes are winning or losing $z > 0$. (Assume he will accept if indifferent.)

(b) Use Taylor's Theorem to derive an approximate expression for p when z is small, and use your expression to show that p is then an increasing function of z.

(c) What aspect of the agent's risk preferences determines whether p is an increasing function of w for small z? Explain.

6. (in memoriam) According to Paul Samuelson, the mathematician Stanislaw Ulam once defined a coward as someone who will not bet even when you offer him two-to-one odds and let him choose his side. (A gamble with two-to-one odds is one in which the individual wins 2x if an event A occurs and loses x if A does not occur. Letting the individual choose his side means letting him choose between winning 2x if A occurs and losing x if A does not occur, or winning 2x if A does not occur and losing x if A occurs.)

(a) Show by example (graphical, if you prefer) that it is possible for an expected utility maximizer who likes money to be a coward according to Ulam's definition.

(b) Show (graphically, if you prefer) that an expected utility maximizer who likes money, and whose von Neumann-Morgenstern utility function is differentiable, cannot be a Ulam-coward for all values of $x > 0$.

7. A risk-averse, state-independent expected utility-maximizing investor who likes money must decide how much of his initial wealth to invest in a risky asset, investing whatever remains in a safe asset. (These are the only two possible investments, and he cannot borrow or lend.) There are two states of the world, with known probabilities, such that 1 invested in the safe asset yields (1+s) no matter which state occurs, and 1 invested in the risky asset yields (1+r) if state 1 occurs and 1 if state 2 occurs.

(a) Suppose that $0 < s < r$. Letting x_i denote the investor's final wealth if state i occurs, graph his opportunity set, given initial wealth I, in (x_1, x_2)-space, (putting x_1 on the horizontal axis). Label your graph clearly to show how I, r, and s, determine the set.

(b) Draw the path in (x_1, x_2)-space that shows how the state-contingent final wealths generated by his optimal portfolio vary with I, when $0 < s < r$ and his Arrow-Pratt measure of absolute risk aversion is constant, independent of his wealth.

(c) Answer part (b) again, but when his relative risk aversion is constant.

(d) How do your answers to (b) and (c) change when $0 < r < s$?

8. Formulate and prove the statement that a first-order stochastically dominating shift in a distribution always increases its mean. Compare the strengths and weaknesses of ordering distributions by first-order stochastic dominance and by their means.

9. A monopolistic firm faces a demand function of the form $D(p,a) \equiv d(p)a$, where p is price and a is a parameter distributed on [0, M] with density f(a). The firm chooses price and quantity to maximize expected profit (it is risk-neutral), but it must choose these variables before the demand parameter a is observed. Suppose that output can be produced at constant marginal cost, c; that unsold output has no value; and that realized demand in excess of the quantity produced cannot be met. Prove that a mean-preserving spread in the distribution of a lowers *maximized* expected profit. (Hint: Write expected profit, for each p and q, as the sum of expected profit given that demand is greater than q, and given that demand is less than q. Then integrate by parts to simplify this expression for the firm's objective function, so that the integral condition can be used to characterize the effect of increasing risk on the firm's problem. Explain each step of your argument.)

10. There are two states of the world, 1 and 2, and a single consumption good; the state-contingent consumption vector $e \equiv (e_1,e_2)$ represents consumption of e_i units of the consumption good if state i occurs. The probability of state i is p_i. Suppose that an individual chooses among state-contingent consumption vectors to maximize the expectation of the state-independent vN-M utility function $u(\cdot)$.

(a) Write the equation of a typical indifference curve for the individual.

(b) Derive an expression for $MRS_{12}(e_1, e_2)$, the individual's marginal rate of substitution between consumption in states 1 and 2 at consumption vector (e_1,e_2).

(c) Suppose that $e^a = (\underline{e}_1,\underline{e}_2)$, $e^b = (\underline{e}_1,\bar{e}_2)$, $e^c = (\bar{e}_1,\underline{e}_2)$, and $e^d = (\bar{e}_1,\bar{e}_2)$, so these vectors form a rectangle in (e_1,e_2)-space. Show that $MRS_{12}(e^a)/MRS_{12}(e^b) = MRS_{12}(e^c)/MRS_{12}(e^d)$.

(d) Does the result of part (c) remain valid when the utility function is state-dependent? Explain.

(e) Does the result of part (c) remain valid when the individual is not an expected-utility maximizer? Explain.