ECON 210B

University of California, San Diego

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Suggested solutions for Q.2 Problem Set 1

Myungkyu Shim

Exercise 2: RBC model and Balanced Growth path

Consider the basic RBC model analyzed in class.

1. Derive the optimality conditions of the social planner problem.

   Solution

   The optimality conditions of the planner's problem are

$$\frac{U_2(C_t, L_t)}{U_1(C_t, L_t)} = \gamma^t A_t F_2(K_t, \gamma^t N_t) \tag{1}$$

$$U_1(C_t, L_t) = \beta E_t[U_1(C_{t+1}, L_{t+1})(A_{t+1}F_1(K_{t+1}, \gamma^t N_t) + 1 - \delta)] \tag{2}$$

$$C_t + K_{t+1} = Y_t + (1-\delta)K_t \tag{3}$$

$$(TVC)\ \lim_{T\to\infty} \beta^T U_1(C_T, L_T)K_{T+1} = 0$$

   where $N_t = 1 - L_t$ and γ^t is the growth rate of the Labor augmenting shock X_t.

2. Show that in a steady state with a constant interest rate, and where consumption grows at a constant rate, the intertemporal elasticity of substitution must be constant. (HINT, consider the consumption Euler equation along the balanced growth path).

3. Show that in order to have a steady state where leisure is constant, and consumption and wages grow at the same rate, the utility function must take the form $U(Cv(L))$. (HINT: show that it must be that $\frac{U_L(C,L)}{U_C(C,L)} = C\frac{U_L(1,L)}{U_C(1,L)}$).

   Solution to question 2 and 3:

   First of all, note that $A = 1$ in the deterministic steady state. We will show that there is a class of utility functions that satisfy Q2 and then giving more restriction to the utility functions in order to satisfy the condition given in Q3.

   Before the analysis, note that because L_t is bounded, it should be constant along the balanced growth path. Then one can show the following is held along the balanced growth path.

$$\gamma_C = \gamma_Y = \gamma_I = \gamma_K = \gamma \text{ and } \gamma_L = \gamma_N = 0$$

   Q2.

Now consider the consumption Euler eq. in the steady state.

$$\underbrace{\frac{U_1(C_t, L)}{U_1(C_{t+1}, L)}}_{(A)} = \beta E(F_1(K, \gamma^t N) + 1 - \delta)] \equiv \Omega$$

The RHS is a constant, thus the term (A) also should be a constant. i.e. the growth rate of marginal utility of consumption should be a constant. Take log to the both sides:

$$\log U_1(C_t, L) = \log U_1(C_{t+1}, L) + \log \Omega$$

Totally differentiate the above equation gives us:

$$\frac{U_{11}(C_t, L)}{U_1(C_t, L)} \Delta C_t = \frac{U_{11}(C_{t+1}, L)}{U_1(C_{t+1}, L)} \Delta C_{t+1} \text{ where } \Delta C_t = C_{t+1} - C_t$$

which means that for the optimality, it requires that

$$\frac{U_{11}(C_t, L)}{U_1(C_t, L)} \Delta C_t = \kappa \text{ (constant) for all } t$$

Note that along the balanced growth path, $C_{t+1} = \gamma C_t$, thus $\Delta C_t = (\gamma - 1) C_t$. Therefore,

$$\frac{U_{11}(C_t, L)}{U_1(C_t, L)} \Delta C_t = \underbrace{\frac{U_{11}(C_t, L)}{U_1(C_t, L)} C_t}_{(B)} (\gamma - 1) = \kappa$$

As $\gamma - 1$ and κ are all constants, it requires that the term (B) also should be a constant. Note that the term (B) is the relative risk aversion, which means that we need CRRA form for consumption. As a result, the utility function is restricted to be as follows in order to satisfy Q3.

$$U(C, L) \begin{cases} = C^{1-\sigma} v^1(L) + v^2(L) \text{ if } \sigma \neq 1 \\ = \ln C \cdot v^1(L) + v^2(L) \text{ if } \sigma = 1 \end{cases} \tag{4}$$

Q3.

Now consider the Labor euler equation. At the steady state, we need

$$\frac{U_2(C_t, L)}{U_1(C_t, L)} = \underbrace{\gamma^t A_t F_2(K_t, \gamma^t N_t)}_{W_t}$$

Note that we have already a restriction on the functional form of the utility function as in (4).

(1) $\sigma \neq 1$

Using the equation (4), we have

$$\frac{C_t^{1-\sigma} v^{1\prime}(L) + v^{2\prime}(L)}{(1-\sigma) C_t^{-\sigma} v^1(L)} = W_t$$

In order to have the constant C_t/W_t(consumption and real wage grows at the same rate), we need $v^{2\prime}(L) = 0$. If not, C_t and W_t cannot have the same growth rate. This means that $V^2(L)$ is a constant, which can be ignored. If $v^{2\prime}(L) = 0$, the above equation is rewritten as follows:

$$\frac{v^{1\prime}(L)}{v^1(L)} = (1-\sigma) \frac{W_t}{C_t}$$

As RHS is constant, L should be constant.

(2) $\sigma = 1$

Again, use the equation (4). Then the Labor Euler equation becomes

$$C_t \frac{\ln C_t \cdot v^{1\prime}(L) + v^{2\prime}(L)}{v^1(L)} = W_t$$

The only way to have constant C_t/W_t is by setting $V^{1\prime}(L) = 0$ which means that $V^1(L)$ is now a constant. If not, $\gamma_c \neq \gamma_w$.

Thus, the only functional form that satisfy Q2 and Q3 together is:

$$U(C, L) \begin{cases} = C^{1-\sigma} v(L) \text{ if } \sigma \neq 1 \\ = \ln C + v(L) \text{ if } \sigma = 1 \end{cases}$$

As a result, one can see that the utility function is the form of $u(Cv(L))$.