ECON 210B

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**Suggested solutions for Problem Set II**

Myungkyu Shim and Davide Debortoli

**Exercise I: Asset Prices and Accuracy of Log-Linearizations**

Consider a pure endowment economy, where the representative agent makes his choices regarding consumption ($C$) and savings ($B$) solving the following problem:

$$\max_{\{C_t,B_t\}_{t=0}^{\infty}} = E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma}$$

$$\text{s.t. } Y_t + B_{t-1} \geq C_t + Q_t B_t,$$

where $Q_t$ is the price of a risk-free discount bond delivering one unit of consumption in period $t+1$. The endowment $Y_t$ is assumed to follow the exogenous process

$$\frac{Y_t}{Y_{t-1}} = \epsilon_t,$$

where $\log \epsilon_t \sim N(0, s^2)$.

1. Derive the first order conditions of the agent's problem and the market clearing condition, showing that in equilibrium the following condition holds:

$$Q_t = \beta E_t \left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma} \tag{1}$$

**Solution:**

The Euler equation is derived by differentiating the Lagrangean with respect to $C_t$ and $B_t$ and combine the resulting expressions. Moreover, market clearing condition yields $C_t = Y_t$ for all $t$. Thus we get eq. (1).

2. What is the value of $Q$ in a perfect foresight (deterministic) steady state?

**Solution:**

In the deterministic steady state, $Y_t = Y_{t+1}$. Let $Q$ be the steady state price level. Then the equation (1) becomes

$$Q = \beta \tag{2}$$

Intuition: Higher $\beta$ means that Households give more weights on the future consumptions, thus willingness to save increases. As a result, demand for the bond increases which makes the price of the bond to increase in order to clear the market.

3. Solve equation (1) to find an exact expression for the bond price $Q_t$ as a function of constants and/or variables that are observable at time $t$ or earlier. [HINT: You may need to use the following relationship: if $X$ follows a log-normal distribution, then $E(X) = Ee^x = e^{E(x)+\frac{1}{2}Var(x)}$].

(a) What is the average value of $Q$? Is it the same as the steady state value computed in 2)? Why?

**Solution:**

From the Euler equation,

$$Q_t = \beta E_t[\exp(-\sigma \underbrace{\log(Y_{t+1}/Y_t)}_{=\ln \varepsilon_{t+1}}) = \beta E_t[\exp(-\sigma \underbrace{\log \varepsilon_{t+1}}_{\sim N(0,s^2)})]$$

Using the property of the log-normal distribution, one can show the following:

$$Q_t = \beta \exp[-\sigma \underbrace{E_t \log \varepsilon_{t+1}}_{=0} + \frac{\sigma^2}{2} \underbrace{var_t(\log \varepsilon_{t+1})}_{=s^2}]$$

As a result,

$$Q_t = \beta \exp[\sigma^2 s^2/2] \tag{3}$$

Since the bond price is constant, its average value is

$$\bar{Q} = Q_t = \beta \exp[\sigma^2 s^2/2]$$

(b) How does the variance of the endowment process ($s^2$) affect the bond price?

**Solution:**

One can easily observe this from the equilibrium price equation (3). As the uncertainty($s^2$) increases, the bond price increases.

The intuition is simple : Precautionary savings motive. As the economy becomes more volatile, households are more willing to buy bonds in order to prepare for the uncertain future. [1] Thus in order to clear the market, the price of the bond should become higher.

(c) Find the equilibrium condition for the interest rate $i_t \equiv -\log Q_t$.

**Solution:**

Take log to the both sides of the equation (3) and multiply by -1 yields:

$$i_t = -\log Q_t = -\log \beta - \frac{\sigma^2 s^2}{2} \tag{4}$$

4. Suppose that neither you nor your computer could find the exact solution obtained in point 3 above. Log-linearize (1) around the steady-state, and obtain an approximate solution for the interest rate as a function of constants and/or variables which are observable at time $t$ or earlier. How does the interest rate depend on the variance of the endowment process according to this approximate relation?

[1] Note that $u'''(c) > 0$ is required to have precautionary savings motive and the given utility function satisfies this requirement.

**Solution:**

Rearranging the Euler equation we get

$$E_t \beta Q_t^{-1} \left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma} - 1 = 0,$$

which can be rewritten as

$$E_t[\beta Q^{-1} e^{-\hat{q}_t - \sigma(\hat{y}_{t+1} - \hat{y}_t)}] - 1 = 0,$$

where for a generic variable $x_t$ we have used the definition $\hat{x}_t \equiv \log X_t - \log X$. Taking a first-order (Taylor) approximation to the above expression around the steady state, we have

$$-\hat{q}_t - \sigma \underbrace{E_t[\hat{y}_{t+1} - \hat{y}_t]}_{=E_t \log \varepsilon_{t+1} = 0} = 0,$$

.

Since $\hat{q}_t = \log Q_t - \log Q = -i_t - \log \beta$ we get

$$i_t = -\log \beta \tag{5}$$

Notice that (4) and (5) are different. This comes from the fact that log-linearization loses the information related to the second moments(or above).

5. Assume that the standard deviation of the endowment process ($s$) could take values in the interval [0.01,0.1]. Comparing the expressions obtained in points 3c and 4, calculate the approximation error (in percentage points), first assuming that the agent is risk-neutral ($\sigma = 0$) and then assuming that the agent has a degree of risk-aversion $\sigma = 5$.

**Solution:**

Given $s \in [.01, .1]$, it is easy to calculate that $s^2 \in [.0001, .01]$.

(1) $\sigma = 0:$ linear utility or risk-neutral

Then (4) and (5) become identical so that in both of the cases,

$$i_t = -\log \beta$$

This is the case because risk-neutral households only consider mean(first moment) for their decision. Thus, if households are risk-neutral, there is no approximation error.

(2) $\sigma = 5:$ risk-averse

Using the given values and the equation (4), one can see that if we do not log-linearize, the equilibrium interest rate should be:

$$i_t \in [-\log \beta - .00125, -\log \beta - .125]$$

While $i_t = -\log \beta$ when we consider the linearized model. As a result, the approximation error is from .125% to 12.5%. Thus as increase of uncertainty($s^2$), the approximation error also increases.

In summary, log-linearization overestimates the interest rate (equivalently, underestimate the price of the bond) because it ignores the presence of uncertainty, and the associated precautionary savings motive, which is a driving force to drive down the interest rate (or increase the price of the bond).

**Exercise II: An RBC models with a trend**

Consider an economy where a representative agents, given an initial stock of capital ($K_0$) chooses consumption ($C_t$), hours worked ($N_t$) and the capital stock($K_t$) solving the following problem:

$$\max_{\{C_t, K_{t+1}, N_t, L_t\}_{t=0}^{\infty}} = E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$$

$$\text{s.t.} \qquad C_t + K_{t+1} \leq W_t N_t + R_t K_t + (1-\delta) K_t$$
$$L_t = 1 - N_t,$$

where $L_t$, is leisure and $W_t$ and $R_t$ indicate the real wage and the real interest rate, respectively. Output is produced by firms operating in a competitive market, according to the production function

$$Y_t = A_t F(K_t, \gamma^t N_t)$$

where $\gamma$ is the growth rate of labor-augmenting technology, and

$$A_t = A_{t-1} exp(\epsilon_t).$$

Answer the following questions:

1. Derive the optimality conditions of households and firms, and provide a complete definition of competitive equilibrium, indicating all the equations that should be satisfied in that equilibrium.

   **Solution:**

   The definition of the competitive equilibrium is omitted from the solution because it is exactly same except that we have one more variable for the exogenous variables $3(\gamma^t)$.

   Assume explicitly that the production function is HD1. Then the optimality conditions for households are described by the following equations:

$$(HH.1)\ \frac{U_2(C_t, L_t)}{U_1(C_t, L_t)} = W_t \tag{6}$$

$$(HH.2)\ U_1(C_t, L_t) = \beta E_t[U_1(C_{t+1}, L_{t+1})(R_{t+1} + 1 - \delta)] \tag{7}$$

$$(HH.BC)\ C_t + K_{t+1} = W_t N_t + R_t K_t + (1-\delta) K_t \tag{8}$$

$$N_t = 1 - L_t \tag{9}$$

$$(TVC)\ \lim_{T \to \infty} \beta^T U_1(C_T, L_T) K_{T+1} = 0$$

   where $U_n$ is the partial derivative with respect to the $n$th argument in the utility function.

   Firm optimality conditions are as follows.

$$(FF.1)\ W_t = \gamma^t A_t F_2(K_t, \gamma^t N_t) \tag{10}$$

$$(FF.2)\ R_t = A_t F_1(K_t, \gamma^t N_t) \tag{11}$$

   In addition, we need the feasibility condition:

$$(FC)\ C_t + K_{t+1} = Y_t + (1-\delta) K_t \tag{12}$$

   One can easily show that if $(FC)$ is met in the equilibrium, so is $(HH.BC)$.

2. Rewrite the equilibrium equations derived in point 1 in terms of variables that are constant in steady state (e.g. replace $Y_t$ with $y_t \equiv \frac{Y_t}{\gamma^t}$).

   **Solution:**

   We know from Q2 that we need a functional form that makes $\frac{U_L(C,L)}{U_C(C,L)} = C\frac{U_L(1,L)}{U_C(1,L)}$. Thus

$$(HH.1)'\ C_t\frac{U_2(1,L)}{U_1(1,L)} = W_t \Leftrightarrow c_t\frac{U_2(1,L)}{U_1(1,L)} = w_t$$

   where $x_t = X_t/\gamma^t$.

$$(HH.2)'\ U_1(\gamma^t c_t, L_t) = \beta E_t[U_1(\gamma^{t+1}c_{t+1}, L_{t+1})[R_{t+1} + 1 - \delta]]$$

   $(HH.BC)$ is now ignored here because it is redundant once we consider $(FC)$.

$$(FF.1)'\ w_t = A_t F_2(k_t, N_t)$$

$$(FF.2)'\ R_t = A_t F_1(k_t, N_t)$$

$$(FC)'\ c_t + \gamma k_{t+1} = A_t F(k_t, N_t) + (1-\delta)k_t$$

   $(FC)'$ is derived by dividing $(FC)$ by $\gamma^t$.

3. Now, consider the following functional forms:

$$U(C_t, L_t) = log(C_t) + \theta\frac{L_t^{1-\eta} - 1}{1-\eta}$$
$$F(K_t, \gamma^t N_t) = K_t^{\alpha}(\gamma^t N_t)^{1-\alpha}$$

   (a) Using the optimality conditions obtained above, show that if $\eta = 1$ and $\delta = 1$ it must be that $C_t/Y_t$ and $L_t$ are constant $\forall t$. Provide an economic intuition for this result.

   **Solution:** Then equilibrium conditions are:

$$(EQ1)\ \frac{\theta C_t}{1-N_t} = (1-\alpha)\underbrace{A_t K_t^{\alpha}\gamma^{(1-\alpha)t}N_t^{-\alpha}}_{=Y_t/N_t}$$

$$(EQ2)\ \frac{1}{C_t} = \beta E_t\left[\frac{1}{C_{t+1}}\alpha\underbrace{A_{t+1}K_{t+1}^{\alpha-1}(\gamma^{t+1}N_{t+1})^{1-\alpha}}_{=Y_{t+1}/K_{t+1}}\right]$$
$$N_t = 1 - L_t$$

$$(FC)\ C_t + K_{t+1} = A_t K_t^{\alpha}(\gamma^t N_t)^{1-\alpha}$$

   $(EQ1)$ is obtained by combining $(HH.1)$ and $(FF.1)$ while $(EQ2)$ comes from combining $(HH.2)$ and $(FF.2)$. One of the main advantage of the assumption that $\eta = 1$(log utility) and $\delta = 1$ (full depreciation) is that we can obtain a closed form solution. Guess the solution form as follows.

$$\begin{aligned} C_t &= = \phi Y_t \\ K_{t+1} &= (1-\phi)Y_t \end{aligned}$$

[Notice that this economy should use output ($Y_t$) for either consumption ($C_t$) or investment ($K_{t+1}$) as can be seen from ($FC$)]. Then in order to verify our guess, we should check that $\phi$ is really a constant. To do so, substitute the above guesses into ($EQ2$)

$$(EQ2)\ \frac{1}{\phi Y_t} = \beta E_t\left[\frac{\alpha}{\phi Y_{t+1}}\frac{Y_{t+1}}{K_{t+1}}\right] \Leftrightarrow \frac{1}{\phi Y_t} = \frac{\alpha\beta}{\phi}\underbrace{\frac{1}{K_{t+1}}}_{=1/(1-\phi)Y_t}$$

where the expectation operator is ignored because $K_{t+1}$ is determined at $t$. Rearranging, we get

$$1-\phi = \alpha\beta, \text{ equivalently, } \phi = 1-\alpha\beta$$

Therefore, our guess is now verified. $\phi$ is constant so that consumption and investment ($I_t = K_{t+1}$) are a fixed fraction of output for all $t$. Thus, under the given parametric assumptions ($\eta = 1$ and $\delta = 1$), it is optimal for households to fix the ratio of consumption to income for any period.

Finally using (EQ1)

$$(EQ1)\ \frac{\theta}{1-N_t}\phi Y_t = (1-\alpha)\frac{Y_t}{N_t}\ \Leftrightarrow \theta\phi N_t = (1-\alpha)(1-N_t)$$

Rearranging, we get

$$N_t = N = \frac{1-\alpha}{\theta\phi + (1-\alpha)}$$

Thus $L_t$ would also be a constant provided that $\phi$ is verifed as a constant. The intuition is that as have seen from 210A, subsitution effect and income effect of wage changes are always same with log utility on leisure (the two opposite effects are canceled out). Therefore, hours worked should be constant even if the marginal product of labor, and thus the wage rate, are fluctuating over time.

In summary,

$$\begin{aligned} C_t &= (1-\alpha\beta)A_t K_t^{\alpha}(\gamma^t N)^{1-\alpha} \\ K_{t+1} &= \alpha\beta A_t K_t^{\alpha}(\gamma^t N)^{1-\alpha} \\ N_t &= N = \frac{1-\alpha}{\theta(1-\alpha\beta)+(1-\alpha)} \end{aligned}$$

Notice that the above equations are the 'policy functions' characterizing the optimal allocations as a function of the state variables.

(b) Log-linearize the equilibrium conditions and write them in the compact form

$$A_0\begin{bmatrix}\hat{k}_t \\ \hat{c}_t\end{bmatrix} = A\begin{bmatrix}\hat{k}_{t+1} \\ E_t\hat{c}_{t+1}\end{bmatrix} + Ba_t,$$

where the variables with hats indicates log deviations from steady state, and $A_0$, $A$ and $B$ are matrices to be determined (feel free to conveniently redefine group of coefficients).

**Solution:**

Equilibrium conditions are as follows(Note that we consider a general case where $\eta \neq 1$ and $\delta \neq 1$).

$$\begin{aligned} (EQ1)\ \frac{\theta C_t}{L_t^{\eta}} &= (1-\alpha)A_t K_t^{\alpha}\gamma^{(1-\alpha)t}N_t^{-\alpha} \\ (EQ2)\ \frac{1}{C_t} &= \beta E_t[\frac{1}{C_{t+1}}(\alpha A_{t+1}K_{t+1}^{\alpha-1}(\gamma^{t+1}N_{t+1})^{1-\alpha}+1-\delta)] \\ L_t &= 1-N_t \\ (FC)\ C_t + K_{t+1} &= A_t K_t^{\alpha}(\gamma^t N_t)^{1-\alpha} + (1-\delta)K_t \end{aligned}$$

Let's transform the variables in order for them to have steady state. As before, $x_t = X_t/\gamma^t$. Then

$$\begin{aligned}(EQ1)\ \frac{\theta c_t}{(1-N_t)^\eta} &= (1-\alpha)A_t k_t^\alpha N_t^{-\alpha} \\ (EQ2)\ \frac{1}{c_t} &= \frac{\beta}{\gamma}E_t[\frac{1}{c_{t+1}}(\alpha A_{t+1}k_{t+1}^{\alpha-1}N_{t+1}^{1-\alpha}+1-\delta)] \\ (FC)\ c_t+\gamma k_{t+1} &= A_t k_t^\alpha N_t^{1-\alpha}+(1-\delta)k_t\end{aligned}$$

Log linearize the above equilibrium conditions around the steady state:

$$(EQ1)\ \frac{\theta c}{(1-N)^\eta}\hat{c}_t+\eta\frac{\theta c}{(1-N)^{\eta+1}}N\hat{n}_t = (1-\alpha)k^\alpha N^{-\alpha}[a_t+\alpha\hat{k}_t-\alpha\hat{n}_t]$$

Using the steady state equation that $\theta c/(1-N)^\eta = (1-\alpha)k^\alpha N^{-\alpha}$,

$$(EQ1)'\ \hat{c}_t+\eta\frac{N}{1-N}\hat{n}_t = a_t+\alpha\hat{k}_t-\alpha\hat{n}_t \Leftrightarrow \hat{n}_t = \frac{1}{\eta\frac{N}{1-N}+\alpha}[a_t+\alpha\hat{k}_t-\hat{c}_t]$$

In addition,

$$(EQ2)\ -\frac{1}{c}\hat{c}_t = \frac{\beta}{\gamma}E_t[-\frac{1}{c}(\alpha k^{\alpha-1}N^{1-\alpha}+1-\delta)\hat{c}_{t+1}+\frac{\alpha}{c}k^{\alpha-1}N^{1-\alpha}(a_{t+1}+(\alpha-1)\hat{k}_{t+1}+(1-\alpha)\hat{n}_{t+1})]$$

Again, use the steady state relationship$(1=\frac{\beta}{\gamma}(\alpha k^{\alpha-1}N^{1-\alpha}+1-\delta))$ so that obtain

$$(EQ2)'\ -\hat{c}_t = E_t[-\hat{c}_{t+1}+(1-\frac{\beta}{\gamma}(1-\delta))(a_{t+1}+(\alpha-1)\hat{k}_{t+1}+(1-\alpha)\hat{n}_{t+1})]$$

Substituting $(EQ1)'$ into $(EQ2)'$,

$$\begin{aligned}(EQ2)'\ \hat{c}_t &= E_t\hat{c}_{t+1}-(1-\frac{\beta}{\gamma}(1-\delta))E_t[a_{t+1}+(\alpha-1)\hat{k}_{t+1}+(1-\alpha)\frac{1}{\eta\frac{N}{1-N}+\alpha}[a_{t+1}+\alpha\hat{k}_{t+1}-\hat{c}_{t+1}]] \\ &= E_t\hat{c}_{t+1}-(1-\frac{\beta}{\gamma}(1-\delta))E_t[(1+\frac{1-\alpha}{\eta\frac{N}{1-N}+\alpha})a_{t+1}+(\alpha-1+\alpha\frac{1-\alpha}{\eta\frac{N}{1-N}+\alpha})\hat{k}_{t+1}-\frac{1-\alpha}{\eta\frac{N}{1-N}+\alpha}\hat{c}_{t+1}] \\ &= \Omega_1 E_t\hat{c}_{t+1}+\Omega_2\hat{k}_{t+1}+\Omega_3 a_t\end{aligned}$$

where $\Omega_1 = 1-(1-\frac{\beta}{\gamma}(1-\delta))\frac{1-\alpha}{\eta\frac{N}{1-N}+\alpha}$, $\Omega_2 = -(1-\frac{\beta}{\gamma}(1-\delta))(1-\alpha)(-1+\frac{\alpha}{\eta\frac{N}{1-N}+\alpha})$, and $\Omega_3 = -(1-\frac{\beta}{\gamma}(1-\delta))(1+\frac{1-\alpha}{\eta\frac{N}{1-N}+\alpha})$.

Finally, let's linearize the feasibility condition.

$$\begin{aligned}(FC)\ c\hat{c}_t+\gamma k\hat{k}_{t+1} &= k^\alpha N^{1-\alpha}(a_t+\alpha\hat{k}_t+(1-\alpha)\hat{n}_t)+(1-\delta)k\hat{k}_t \\ &= k^\alpha N^{1-\alpha}(a_t+\alpha\hat{k}_t+(1-\alpha)\frac{1}{\eta\frac{N}{1-N}+\alpha}[a_t+\alpha\hat{k}_t-\hat{c}_t])+(1-\delta)k\hat{k}_t\end{aligned}$$

Rearranging, we get

$$(FC)'\ \Omega_4\hat{c}_t+\gamma k\hat{k}_{t+1} = \Omega_5 a_t+\Omega_6\hat{k}_t$$

where $\Omega_4 = c+\frac{k^\alpha N^{1-\alpha}(1-\alpha)}{\eta\frac{N}{1-N}+\alpha}$, $\Omega_{5=}k^\alpha N^{1-\alpha}(1+\frac{1-\alpha}{\eta\frac{N}{1-N}+\alpha})$, and $\Omega_6 = \alpha k^\alpha N^{1-\alpha}(1+\frac{1-\alpha}{\eta\frac{N}{1-N}+\alpha})+(1-\delta)k = \alpha\Omega_5+(1-\delta)k$.

In summary, the log-linearized system is:

$$\begin{aligned}(EQ2)'\ \hat{c}_t &= \Omega_1 E_t\hat{c}_{t+1}+\Omega_2\hat{k}_{t+1}+\Omega_3 a_t \\ (FC)\ \Omega_4\hat{c}_t-\Omega_6\hat{k}_t &= \Omega_5 a_t-\gamma k\hat{k}_{t+1}\end{aligned}$$

Then one can express the above system in a matrix form equation.

$$\begin{bmatrix} 0 & 1 \\ -\Omega_6 & \Omega_4 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} = \begin{bmatrix} \Omega_2 & \Omega_1 \\ -\gamma k & 0 \end{bmatrix} \begin{bmatrix} \hat{k}_{t+1} \\ E_t \hat{c}_{t+1} \end{bmatrix} + \begin{bmatrix} \Omega_3 \\ \Omega_5 \end{bmatrix} a_t,$$

(c) Indicate the conditions that $A$ and $A_0$ should satisfy for existence and uniqueness of the solution.

**Solution:**

First of all, the determinant of $A_0$ is $\Omega_6$.i.e. $\det(A_0) = \Omega_6 > 0$. Thus it is invertible so that we can define $\Phi = A_0^{-1}A$. As $\Phi$ is a 2 by 2 matrix, it has two eigenvalues. Then in order for the existence and the uniqueness of the solution, it is required that the number of the eigenvalues which are inside the unit circle is equal to the number of control variables appearing in the system. Here, we have only one control variable ($\hat{c}_t$), so we must have ONE eigenvalue within the unit circle.

(d) Using the method of undetermined coefficients find the solution of the model. In particular, guess that the solution takes the form

$$\begin{bmatrix} \hat{k}_{t+1} \\ \hat{c}_t \end{bmatrix} = \begin{bmatrix} \psi_{kk} \\ \psi_{ck} \end{bmatrix} \hat{k}_t + \begin{bmatrix} \psi_{ka} \\ \psi_{ca} \end{bmatrix} a_t$$

and solve for the coefficients $\psi_{kk}$, $\psi_{ck}$, $\psi_{ka}$ and $\psi_{ca}$

**Solution:**

Following the steps shown during the lecture. Substitute the guess into the system of the equations, and find the values of the guess by matching the coefficients.