ECON 210B

University of California, San Diego

Winter 2012

Problem Set 2

(due on Fri. Jan 27^{th})

Exercise I: Asset Prices and Accuracy of Log-Linearizations

Consider a pure endowment economy, where the representative agent makes his choices regarding consumption (C) and savings (B) solving the following problem:

$$\max_{\{C_t,B_t\}_{t=0}^{\infty}} = E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma}$$

$$\text{s.t. } Y_t + B_{t-1} \geq C_t + Q_t B_t,$$

where Q_t is the price of a risk-free discount bond delivering one unit of consumption in period $t+1$. The endowment Y_t is assumed to follow the exogenous process

$$\frac{Y_t}{Y_{t-1}} = \epsilon_t,$$

where $\log \epsilon_t \sim N(0, s^2)$.

1. Derive the first order conditions of the agent's problem and the market clearing condition, showing that in equilibrium the following condition holds:

$$Q_t = \beta E_t \left(\frac{Y_{t+1}}{Y_t} \right)^{-\sigma} \tag{1}$$

2. What is the value of Q in a perfect foresight (deterministic) steady state?

3. Solve equation (1) to find an exact expression for the bond price Q_t as a function of constants and/or variables that are observable at time t or earlier. [HINT: You may need to use the following relationship: if X follows a log-normal distribution, then $E(X) = Ee^x = e^{E(x)+\frac{1}{2}Var(x)}$].

   (a) What is the average value of Q? Is it the same as the steady state value computed in 2)? Why?

   (b) How does the variance of the endowment process (s^2) affect the bond price?

   (c) Find the equilibrium condition for the interest rate $i_t \equiv -\log Q_t$.

4. Suppose that neither you nor your computer could find the exact solution obtained in point 3 above. Log-linearize (1) around the steady-state, and obtain an approximate solution for the interest rate as a function of constants and/or variables which are observable at time t or earlier. How does the interest rate depend on the variance of the endowment process according to this approximate relation?

5. Assume that the standard deviation of the endowment process (s) could take values in the interval [0.01,0.1]. Comparing the expressions obtained in points 3c and 4, calculate the approximation error (in percentage points), first assuming that the agent is risk-neutral ($\sigma = 0$) and then assuming that the agent has a degree of risk-aversion $\sigma = 5$.

Exercise II: An RBC models with a trend

Consider an economy where a representative agents, given an initial stock of capital (K_0) chooses consumption (C_t), hours worked (N_t) and the capital stock(K_t) solving the following problem:

$$\max_{\{C_t, K_{t+1}, N_t, L_t\}_{t=0}^{\infty}} = E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$$

$$\text{s.t.} \qquad C_t + K_{t+1} \leq W_t N_t + R_t K_t + (1-\delta) K_t$$
$$L_t = 1 - N_t,$$

where L_t, is leisure and W_t and R_t indicate the real wage and the real interest rate, respectively. Output is produced by firms operating in a competitive market, according to the production function

$$Y_t = A_t F(K_t, \gamma^t N_t)$$

where γ is the growth rate of labor-augmenting technology, and

$$A_t = A_{t-1} exp(\epsilon_t).$$

Answer the following questions:

1. Derive the optimality conditions of households and firms, and provide a complete definition of competitive equilibrium, indicating all the equations that should be satisfied in that equilibrium.

2. Rewrite the equilibrium equations derived in point 1 in terms of variables that are constant in steady state (e.g. replace Y_t with $y_t \equiv \frac{Y_t}{\gamma^t}$).

3. Now, consider the following functional forms:

$$U(C_t, L_t) = log(C_t) + \theta \frac{L_t^{1-\eta} - 1}{1-\eta}$$
$$F(K_t, \gamma^t N_t) = K_t^{\alpha} (\gamma^t N_t)^{1-\alpha}$$

   (a) Using the optimality conditions obtained above, show that if $\eta = 1$ and $\delta = 0$ it must be that C_t/Y_t and L_t are constant $\forall t$. Provide an economic intuition for this result.

   (b) Log-linearize the equilibrium conditions and write them in the compact form

$$A_0 \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} = A_1 \begin{bmatrix} \hat{k}_{t+1} \\ E_t \hat{c}_{t+1} \end{bmatrix} + B_0 a_t,$$

   where the variables with hats indicates log deviations from steady state, and A_0, A_1 and B_0 are matrices to be determined (feel free to conveniently redefine group of coefficients).

   (c) Indicate the conditions that A_1 and A_0 should satisfy for existence and uniqueness of the solution.

   (d) Using the method of undetermined coefficients find the solution of the model. In particular, guess that the solution takes the form

$$\begin{bmatrix} \hat{k}_{t+1} \\ \hat{c}_t \end{bmatrix} = \begin{bmatrix} \psi_{kk} \\ \psi_{ck} \end{bmatrix} \hat{k}_t + \begin{bmatrix} \psi_{ka} \\ \psi_{ca} \end{bmatrix} a_t$$

   and solve for the coefficients ψ_{kk}, ψ_{ck}, ψ_{ka} and ψ_{ca}.