ECON 210B
The Basic Real Business Cycle Model

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Overview

1. The RBC model

2. Solution methods
    2.1 The perturbation method ()
    2.2 ... Existence and uniqueness: The Blanchard - Kahn conditions
    2.3 ... The method of undetermined coefficients

3. Comparing the model with the data

The RBC model: main ingredients

- Neo-Classical Growth Model (Ramsey - Cass - Koopmans)
  ... Infinite horizon, consumption smoothing +
    1. Shocks (Technology Shocks)
    2. Labor - Leisure choice
- Parsimonious model: No money, No heterogeneity
- ... but a good starting point, at least for methodology.

The RBC model: technology and constraints

notation follows King and Rebelo (2000)

- *Technology*

$$Y_t = A_t F(K_t, X_t N_t)$$

with $F(\cdot)$ strictly concave, twice continuously differentiable, HD1, and satisfying the Inada conditions.

  - A_t is the "stochastic" productivity shock
  - $X_t = \gamma X_{t-1}$ is the "deterministic" (Harrod-neutral) productivity shock

- *Constraints and Initial conditions*

$$\begin{aligned} N_t + L_t &= 1 \\ Y_t &\geq C_t + I_t \\ K_{t+1} &= (1-\delta)K_t + I_t \end{aligned}$$

and A_0, K_0 given, $X_0 = 1$.

The RBC model: the utility function

- *Preferences:*

  $$E_0 \sum_{t=0}^{\infty} b^t u(C_t, L_t),$$

  $u(\cdot)$ is concave, twice continuously differentiable

- ... + other regularity conditions for balanced growth path [see King, Plosser and Rebelo (JME 1988)]

  $$U(C_t, 1-N_t) = \frac{[C_t v(1-N_t)]^{1-\sigma} - 1}{1-\sigma}$$

The RBC model: the social planner's problem

(after transforming the problem)

- Social planner solution ⇔ Competitive Equilibrium
- ... let's then solve the social planner problem (easier), and then look at the decentralization.
- Given our assumptions on technology (HD1) and preferences

$$
\begin{array}{rl}
\max\limits_{\{c_t, N_t, k_{t+1}\}_{t=0}^{\infty}} & E_0 \sum\limits_{t=0}^{\infty} \overbrace{\beta^t}^{[b\gamma^{1-\sigma}]^t} \dfrac{[cv(1-N)]^{1-\sigma}-1}{1-\sigma} \\
s.t. & A_t F(k_t, N_t) + (1-\delta)k_t \geq c_t + \gamma k_{t+1} \\
 & c_t > 0,\ 0 < N_t < 1 \text{ and } k_0,\ A_0 \text{ given}
\end{array}
$$

with $c_t \equiv C_t / X_t$ and $k_t \equiv K_t / X_t$. and where the law of motion of A_t is also known.

The RBC model: the social planner's problem (cont'd)

More compactly the problem is thus

$$\begin{aligned} \max_{\{c_t, N_t, k_{t+1}\}_{t=0}^{\infty}} \quad & E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - N_t) \\ s.t. \quad & A_t F(k_t, N_t) + (1-\delta) k_t \geq c_t + \gamma k_{t+1} \end{aligned}$$

The optimality conditions (e.g. from Lagrangean) are

$$\begin{aligned} & -\frac{u_{N,t}}{u_{c,t}} = A_t F_{N,t} \\ & u_{c,t} = \beta E_t u_{c,t+1} \left[1 - \delta + A_{t+1} F_{k,t+1} \right] \\ & A_t F(k_t, N_t) + (1-\delta) k_t = c_t + \gamma k_{t+1} \\ & \lim_{t \to \infty} \beta^t u_{c,t} F_{k,t+1} = 0 \end{aligned}$$

NOTE: the conditions must hold $\forall t = 0, 1, ...$, and for any history of the shocks $\{A_t\}$

EXERCISE: Show that the above conditions coincide the optimality conditions of a competitive equilibrium.

Solving the model

- What does it mean?
  - Given the initial conditions and a stochastic process for the shocks (exogenous)
  - ... find the sequence $\{c_t, N_t, k_{t+1}\}_{t=0}^{\infty}$ (endogenous)

- Main challenge: infinitely many variables to be solved for
  - ... infinite horizon
  - ... infinitely many histories

- Approach: Recursive Methods

Recursive methods: an intuition

In $t = 0$, A_0 and k_0 are known
... need to find $\{c_0, N_0, k_1\} \Rightarrow$ If we succeed we have k_1

In $t = 1$, A_1 and k_1 are known
... need to find $\{c_1, N_1, k_2\} \Rightarrow$ If we succeed we have k_2.

... and so on

MAIN IDEA: the mapping $\{A_t, k_t\} \longrightarrow \{c_t, N_t, k_{t+1}\}$ is the SAME for all t
i.e. a Time - Invariant function $f(\cdot)$

$$\begin{bmatrix} c_t \\ N_t \\ k_{t+1} \end{bmatrix} = f\left(k_t, A_t\right)$$

Instead of solving for the (infinite-dimensional) sequence $\{c_t, N_t, k_{t+1}\}_{t=0}^{\infty}$,
we can just solve for ONE function $f(\cdot)$.

Recursive methods: some terminology

$$\left[\begin{array}{c} x_t \\ s_{t+1} \end{array}\right] = f\left(s_t, \epsilon_t\right)$$

$$\left[\begin{array}{c} c_t \\ N_t \\ k_{t+1} \end{array}\right] = f\left(k_t, A_t\right)$$

In general	In this example
Policy function: $f(\cdot)$	
State (pre-dertemined) variables	
Endogenous: s_t	k_t
Exogenous: ϵ_t	A_t
Control (jump) variables: x_t, s_{t+1}	c_t, N_t, k_{t+1}

Recursive methods: How to find $f(\cdot)$?

Many available methods:

1. Value function iteration
2. Projection methods
3. Perturbartion methods

Technologies:

a. Paper, pencil (... and good luck!)

b. A computer (for specific functions and parameters)

The Perturbation method

1. Find the steady state (... existence, uniqueness? ... DONE in 210A)
2. Take a (log) linear approximation of the FOC around that steady state we have a system of equations

$$A_0 \begin{bmatrix} x_t \\ s_t \end{bmatrix} = A_1 \begin{bmatrix} E_t x_{t+1} \\ s_{t+1} \end{bmatrix} + B_0 \epsilon_t$$

   where A_0, A_1 and B_0 are functions of the model parameters.

3. Solve the system of linear equations
   1. Check (local) existence and uniqueness (Blanchard - Kahn conditions)
   2. Find the solution: two approaches
      - Method of undetermined coefficients (see H. Uhlig)
      - Generalized Schur decomposition (see C. Sims)

... the solution takes the form $\begin{bmatrix} x_t \\ s_{t+1} \end{bmatrix} = \Psi_s s_t + \Psi_\epsilon \epsilon_t$

.. or more compactly $\underset{[x_t, s_{t+1}]}{z_t} = \underset{[0\ \ \Psi_s]}{\Psi_z} z_{t-1} + \Psi_\epsilon \epsilon_t$

EXERCISE: What is the dimension of Ψ_s and Ψ_ϵ in the RBC model considered earlier?

Checking (local) existence and uniqueness of a stable solution

The Blanchard-Kahn conditions

We have the system

$$A_0 \left[\begin{array}{c} x_t \\ s_t \end{array} \right] = A_1 \left[\begin{array}{c} E_t x_{t+1} \\ s_{t+1} \end{array} \right] + B_0 \epsilon_t$$

Say the size of x_t is n_x, and the size of s_t is n_s

Let's assume that A_0 is invertible (... not a restrictive assumption)
... compute $\lambda \equiv eig\left(A_0^{-1} A_1\right)$
... and denote with k the number of elements of λ strictly less than 1.

A (locally) stable solution:

- it exists iff $k \leq n_x$.
- it is unique iff $k = n_x$

Intuition: Univariate case
EXERCISE: what if $n_s = 0$?

The method of undetermined coefficients

We have the system

$$A_0 \begin{bmatrix} x_t \\ s_t \end{bmatrix} = A_1 \begin{bmatrix} E_t x_{t+1} \\ s_{t+1} \end{bmatrix} + B_0 \epsilon_t \qquad (1)$$

and the solution must take the form

$$\begin{bmatrix} x_t \\ s_{t+1} \end{bmatrix} = \Psi_s s_t + \Psi_\epsilon \epsilon_t \qquad (2)$$

where Ψ_s and Ψ_ϵ have to be determined (i.e. they are our unknowns.)

The procedure:

1. Use (2) to replace for x_t, x_{t+1} and s_{t+1} in (1):

$$E_t x_{t+1} = \Psi_{sx} s_{t+1} \qquad s_{t+1} = \Psi_{ss} s_t + \Psi_{\epsilon s} \epsilon_t \qquad x_t = \Psi_{sx} s_t + \Psi_{\epsilon x} \epsilon_t$$

$$\Rightarrow A_0 \begin{bmatrix} \Psi_{sx} \\ 1 \end{bmatrix} s_t + A_0 \begin{bmatrix} \Psi_{\epsilon x} \\ 0 \end{bmatrix} \epsilon_t = A_1 \begin{bmatrix} \Psi_{sx} \\ 1 \end{bmatrix} \Psi_{ss} s_t + \left(A_1 \begin{bmatrix} \Psi_{sx} \\ 1 \end{bmatrix} \Psi_{\epsilon s} + B_0 \right) \epsilon_t$$

2. Find the (unknown) coefficients solving the (non-linear) system:

$$A_0 \begin{bmatrix} \Psi_{sx} \\ 1 \end{bmatrix} = A_1 \begin{bmatrix} \Psi_{sx} \\ 1 \end{bmatrix} \Psi_{ss} \qquad A_0 \begin{bmatrix} \Psi_{\epsilon x} \\ 0 \end{bmatrix} = \left(A_1 \begin{bmatrix} \Psi_{sx} \\ 1 \end{bmatrix} \Psi_{\epsilon s} + B_0 \right)$$

Comparing the model with the data

1. Calibrate the model (functional forms, parameters)
2. Solve the model, get $z_t = \Psi_z z_{t-1} + \Psi_\epsilon \epsilon_t$, compare some statistics
    2.1 Second moments: variances / co-variances at different lags j

$$\Gamma(j) \equiv E(z_t z_{t-j}^T) = \Psi_z E(z_{t-1} z_{t-j}^T) + \Psi_\epsilon \underbrace{E(\epsilon_t z_{t-j}^T)}_{=0} = (\Psi_z)^j \Gamma(0)$$

$$\Gamma(0) = E(z_t z_t^T) \Rightarrow vec(\Gamma(0)) = (I - \Psi_z \otimes \Psi_z)^{-1} vec \underbrace{(\Sigma)}_{E(\epsilon_t \epsilon_t^T)}$$

    2.2 Impulse responses:

$$z_j = (\Psi_z)^j \, \Psi_\epsilon \epsilon_0$$

… in the data the impulse responses can be obtained from a Vector-Auto Regression (VAR)

Calibration of our RBC model

Functional forms:

Utility function:	$\frac{[c_t v(1-N_t)]^{1-\sigma}-1}{1-\sigma} \log(c_t) + \theta \log(1 - N_t)$	$\sigma = 1$
Production function:	$Y_t = A_t k_t^{\alpha} N_t^{1-\alpha t}$	$\alpha = 1 - \frac{wN}{Y} = .33$

In steady state:

Consumption Euler eq.	$\beta = (1 + R - \delta)^{-1}$	$\delta = 10\%$ $\beta = .984$ (return 6.5%)
Labor Euler eq.	$\theta \frac{c}{1-N} = (1-\alpha)\frac{Y}{N}\ \theta \frac{N}{1-N} = (1-\alpha)\ \underbrace{\frac{Y}{c}}_{1.5}$	θ such that $N = .2$

Technology shocks:

From the data:	$\log Y_t = \alpha \log k_t + (1-\alpha) \log N_t + \log A_t$	$\log A_t$: (Solow) residual
Regress the residual:	$\log A_t = \rho \log A_{t-1} + u_t$	ρ, σ_u^2

RBC: model vs data

from King and Rebelo (1999)

	Relative st. dev.		$Corr(x_t, y_t)$	
	Model	Data	Model	Data
Output (y)	1	1	1	1
Consumption (c)	.44	.74	.94	.88
Investment (I)	2.95	2.93	.99	.80
Hours Worked (N)	.48	.99	.97	.88
Labor productivity (Y/N)	.54	.56	.98	.55
Real Wages (w)	.54	.38	.98	.12
Real Interest Rate (r)	.04	.16	.95	-.35
Total Factor Productivity (A)	.68	.54	1	.78