Extra material to "Fiscal Policy under Loose Commitment" by Davide Debortoli and Ricardo Nunes, Journal of Economic Theory.

The planner's objective in the endogenous probability case (without aggregate shocks)

We show how to obtain the planner's objective function (20). Similarly to the derivations for the exogenous probability case of section 3.2, there are three key steps.

First, notice that the partition of events considered in Figure 1 still applies, as described in the main text. Given the definition of the sets Ω^t_{ND} and $\Omega^t_{D,i}$ we have that by construction $\{\Omega^t_{ND}, \Omega^t_{D,1}, ..., \Omega^t_{D,t}\}$ is a partition of the set Ω^t. Hence, eq. (A-1) can still be obtained.

Second, rewrite $Prob(\omega^t)$ as

$$Prob(\omega^t) = Prob(\omega^i_{D,i} \wedge \omega^t) = Prob(\omega^t | \omega^i_{D,i}) Prob(\omega^i_{D,i}), \forall\ \omega^t \in \Omega^t_{D,i},\ t \geq i$$

and collecting terms, eq. (A-2) is derived.

Third, define the value functions $\xi_i(k_i(\omega^i_{D,i}))$ as in (A-3). Since, $\Omega^t_{D,i} \cap \Omega^t_{D,j} = \oslash$ for $i \neq j$, the choices of different planners are still independent between themselves. Hence, substituting the expression for $\xi(\cdot)$ and plugging in the probability of events

$$\begin{aligned} Prob(\{\omega^t, ND\} | \omega^t) &= P(k_{t+1}) \\ Prob(\{\omega^t, D\} | \omega^t) &= 1 - P(k_{t+1}) \\ Prob(\omega^t_{ND}) &= \frac{\Pi^t_{j=0}(P(k_j))}{P(k_0)} \\ Prob(\omega^t_{D,t}) &= \frac{\Pi^{t-1}_{j=0}(P(k_j))}{P(k_0)} (1 - P(k_t)) \end{aligned}$$

eq. (20) is obtained.

Loose commitment with exogenous aggregate shocks

The purpose of this note is to show that the loose commitment setting proposed in the main text is also valid in the presence of exogenous aggregate shocks. We first consider the case where the probability of commitment is exogenous. At the end, we consider the endogenous probability case.

Suppose the occurrence of Default or No Default is driven by a Markov stochastic process $\{s_t\}_{t=1}^{\infty}$ with possible realizations $\bar{s}_t \in \Phi \equiv \{D, ND\}$. The occurence of productivity shocks is driven by a stochastic process $\{z_t\}_{t=0}^{\infty}$. We refer to $\omega^t \equiv \{s^t, z^t\}$ as a history of events where s^t and z^t summarize the history of events regarding the default shock and the productivity shock, respectively.

Let Γ^t be the set of possible productivity histories up to time t (z^t), and Ω^t be the set of possible histories up to time t:

$$\Omega^t \equiv \{\omega^t = \{D, \{\bar{s}_j\}_{j=1}^t, \{\bar{z}_j\}_{j=0}^t\} : \bar{s}_j \in \Phi, \bar{z}_j \in \Theta,\ \forall j = 0, ..., t\} \tag{1}$$

As in the paper, we only consider the histories $\omega^t = \{D, \bar{s}_1, \bar{s}_2, ..., \bar{s}_t, z^t\}$, i.e. histories that start with a default on past promises. This is because in the initial period there are no promises to be fulfilled or equivalently the current government has just been settled.

We now define the history where no-default has occurred up to time t and the particular history of productivity shocks $z^t = \{\bar{z}_j\}_{j=0}^t$ has been realized as

$$\omega_{ND}^{t,z^t} \equiv \{\omega^t = \{D, \{\bar{s}_j\}_{j=1}^t, z^t\} : \bar{s}_j = ND, \forall j = 1, ..., t\},$$

and the subset of Ω^t collecting all these histories

$$\Omega_{ND}^t \equiv \{\omega^t = \{D, \{\bar{s}_j\}_{j=1}^t, z^t\} : \bar{s}_j = ND, \forall j = 1, ..., t,\ \forall z^t \in \Gamma^t\} = \bigcup_{z^t \in \Gamma^t} \omega_{ND}^{t,z^t}.$$

In an analogous way, we define the set containing the histories where the first default occurs in period i and the particular history of productivity shocks z^t has been realized as

$$\Omega_{D,i}^{t,z^t} \equiv \{\omega^t = \{D, \{\bar{s}_j\}_{j=1}^t, z^t\} : (\bar{s}_i = D) \wedge (\bar{s}_j = ND), \forall j = 1, ..., i-1\}, \text{ if } i \leq t$$
$$\Omega_{D,i}^{t,z^t} \equiv \oslash, \text{ if } i > t,$$

and the set collecting all these histories

$$\Omega_{D,i}^t = \bigcup_{z^t \in \Gamma^t} \Omega_{D,i}^{t,z^t}$$

By construction $\{\Omega_{ND}^t, \Omega_{D,1}^t, ..., \Omega_{D,t}^t\}$ is a partition of the set Ω^t. Using this notation we can write an analogous equation to eq. (A-1) in the main part of the text

$$
\begin{aligned}
W(k_0) = \max_{\substack{\{x_t(\omega^t)\}_{t=0}^{\infty} \\ \omega^t \in \Omega^t}} [\sum_{t=0}^{\infty} \sum_{\omega^t_{ND}} \beta^t \{Prob(\omega^t) u(x_t(\omega^t), k_t(\omega^t))\} + \\
+ \sum_{z^1 \in \Gamma^1} \max_{\substack{\{x_t(\omega^t)\}_{t=1}^{\infty} \\ \omega^t \in \Omega_{D,1}^{t,\{z^1,z^t\}}}} \left\{ \sum_{t=1}^{\infty} \sum_{\omega^t \in \Omega_{D,1}^{t,\{z^1,z^t\}}} \beta^t \{Prob(\omega^t) u(x_t(\omega^t), k_t(\omega^t))\} \right\} \\
+ \sum_{z^2 \in \Gamma^2} \max_{\substack{\{x_t(\omega^t)\}_{t=2}^{\infty} \\ \omega^t \in \Omega_{D,2}^{t,\{z^2,z^t\}}}} \left\{ \sum_{t=2}^{\infty} \sum_{\omega^t \in \Omega_{D,2}^{t,\{z^2,z^t\}}} \beta^t \{Prob(\omega^t) u(x_t(\omega^t), k_t(\omega^t))\} \right\} \\
+ ...]
\end{aligned}
$$

The main difference from eq. (A-1) is that in each $t \geq 1$ we have a summation for each $z^t \in \Gamma^t$. For instance, at $t = 1$ while before there was only one node with a reoptimization, in the present formulation we would have two nodes corresponding to productivity being high or low. These are the nodes that we are summing over.

From now on, the same step as in (A-2)-(A-3) can be followed to obtain the expression

$$
\begin{aligned}
W(k_0) = \max_{\substack{\{x_t(\omega^t)\}_{t=0}^{\infty} \\ \omega^t \in \Omega^t_{ND}}} \sum_{t=0}^{\infty} \beta^t \Bigg\{ \sum_{z^t \in \Gamma^t} Prob(\omega_{ND}^{t,z^t}) u(x_t(\omega_{ND}^{t,z^t}), k_t(\omega_{ND}^{t,z^t})) \\
+ \sum_{i=1}^{\infty} \beta^i \sum_{z^i \in \Gamma^i} Prob(\omega_{D,i}^{i,z^i}) \xi(k_i(\omega_{D,i}^{i,z^i})) \Bigg\}
\end{aligned}
$$

which is analogous to eq. (16) in the text. We have thus separated the histories where the current planner optimizes. In particular, to obtain the above expression we made use of the fact that

$$
Prob(\omega^t) = Prob(\omega_{D,i}^{i,z^i} \wedge \omega^t) = Prob(\omega^t | \omega_{D,i}^{i,z^i}) Prob(\omega_{D,i}^{i,z^i}), \forall\ \omega^t \in \Omega_{D,i}^{t,z^t},\ t \geq i,
$$

Note that $\Omega_{D,t}^{t,z^t}$ is a singleton, so that we can denote it by $\omega_{D,t}^{t,z^t}$. Also,

$$
\xi(k_i(\omega_{D,i}^{i,z^i}), z^i) \equiv \max_{\substack{\{x_t(\omega^t)\}_{t=i}^{\infty} \\ \omega^t \in \Omega_{D,i}^{t,\{z^i,z^t\}}}} \sum_{t=i}^{\infty} \sum_{\omega^t \in \Omega_{D,i}^{t,\{z^i,z^t\}}} \beta^{t-i} Prob(\omega^t | \omega_{D,i}^{i,z^i}) u(x_t(\omega^t), k_t(\omega^t))
$$

The value functions $\xi(k_i(\omega_{D,i}^{i,z^i}), z^i)$ summarize the happenings after the node $\omega_{D,i}^{i,z^i}$. Since $\Omega_{D,i}^{t,z^t} \cap \Omega_{D,j}^{t,z^t} = \oslash$ for $i \neq j$, the plans made at future dates are independent between themselves.

Endogenous probability case with aggregate exogenous shocks

To show that the loose commitment formulation is also valid in the endogenous probability case, notice the above expressions are perfectly compatible with the probability of defaulting depending on state variables. In particular, for any history $\omega^t = (s^t, z^t)$ we have

$$Prob(\omega^t) = Prob(s^t \wedge z^t) = Prob(s^t|z^t)Prob(z^t).$$

In the endogenous probability case $Prob(s^t|z^t)$ depends on the history of capital stocks chosen by the planner conditional on the fact that the history of productivity shocks z^t has occurred, namely

$$Prob(s^t|z^t) = \frac{\Pi_{j=0}^{t}[P(k_j(\omega^j))]}{P(k_0)}.$$

The realizations of Default or No-Default thus depend on the realizations of the productivity shocks. The formula above takes this into account since the choice of capital is conditional on the particular realization of the shocks.[1]

We can thus obtain

$$\begin{aligned} Prob(\omega_{ND}^{t,z^t}) &= \frac{\Pi_{j=0}^{t}(P(k_j))}{P(k_0)} Prob(z^t) \\ Prob(\omega_{D,t}^{t,z^t}) &= \frac{\Pi_{j=0}^{t-1}(P(k_j))}{P(k_0)} \left(1 - P(k_t)\right) Prob(z^t). \end{aligned}$$

To further clarify the issue, we now provide an example. Consider that the productivity shocks follow a two-state Markov chain with:

$$\begin{aligned} Prob(z_{t+1} = H|z_t = H) = q \\ Prob(z_{t+1} = L|z_t = L) = q \end{aligned}$$

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[1] In the above expression, the division by $P(k_0)$ is just a normalization to assign probability 1 to the initial node.

As before, we refer to $\omega^t \equiv \{s^t, z^t\}$ as an history of events. s^t refers to the history of $s_t \in \{Default, NoDefault\}$, and z_t refers to the history of productivity shocks. For instance, an example of an history up to period 2 is

$$\omega^2 = \{s_0 = ND, s_1 = ND, s_2 = D, z_0 = H, z_1 = H, z_2 = L\}$$

In this case,

$$\begin{aligned} s^2 &= \{s_0 = ND, s_1 = ND, s_2 = D\} \\ z^2 &= \{z_0 = H, z_1 = H, z_2 = L\} \end{aligned}$$

Notice that a given history ω^t as a two component $\{s^t, z^t\}$ is always well-defined, even in the case of an endogenous probability event. In any node, we always have a specific realization of s^t and z^t. In order to define the planner problem we need to make sure that we would be able to refer to the probability of reaching each node of the possible history of events. In particular,

$$\begin{aligned} Prob(\omega^2) =& Prob(s^2 \wedge z^2) = Prob(s^2|z^2)Prob(z^2) = \\ &= P(k_1(ND, H))[1 - P(k_2(ND, ND, H, H))]q(1-q). \end{aligned}$$

The issue is that k_{t+1} at every node is a choice variable, conditional on the history of shocks. The planner can choose $k_{t+1}(\omega^{t+1})$ and determine the probability of commitment and default. In this way we can distinguish every node and write the probability of reaching each node as a function of the decisions of the planner.