Econ 220B
James Hamilton

Final Exam
Winter 2004

DIRECTIONS: No books or notes of any kind are allowed. Answer all questions on separate paper. 250 points are possible on this exam

1.) (25 points total) State (but do not prove) the Gauss-Markov Theorem.

2.) (30 points total) Consider the following regression model:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

where $\mathbf{y}$ is a $(T \times 1)$ vector, $\mathbf{X}$ is a $(T \times k)$ matrix, $\boldsymbol{\varepsilon}$ is a $(T \times 1)$ vector with the property that $\boldsymbol{\varepsilon}|\mathbf{X} \sim N(\mathbf{0}, \boldsymbol{\sigma}^2\mathbf{I}_T)$. Suppose you wanted to test the single nonlinear hypothesis $g(\boldsymbol{\beta}) = 0$. Give the formula for a statistic you could use to test this hypothesis and state (but do not prove) the distribution you would use to interpret this statistic.

3.) (40 points total) Consider the following regression, which includes a lagged dependent variable:

$$y_t = \alpha + \beta y_{t-1} + \gamma z_t + \varepsilon_t.$$

Let $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_T)'$ be the $(T \times 1)$ vector containing all the epsilons where T is the sample size and let $\mathbf{z} = (z_1, z_2, ..., z_T)'$. Suppose that $\boldsymbol{\varepsilon}|\mathbf{z} \sim N(\mathbf{0}, \sigma^2\mathbf{I}_T)$ and $y_0 = 0$. Suppose further that the true value of β satisfies $0 < \beta < 1$ and that $\{z_t, y_t\}$ are stationary and ergodic with

$$E\begin{bmatrix} 1 \\ y_{t-1} \\ z_t \end{bmatrix}\begin{bmatrix} 1 & y_{t-1} & z_t \end{bmatrix} = \mathbf{Q}$$

where $\mathbf{Q}$ has rank 3 with all elements of $\mathbf{Q}$ strictly positive. Let $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\gamma}$ be the OLS estimates defined by

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\gamma} \end{bmatrix} = \begin{bmatrix} T & \Sigma y_{t-1} & \Sigma z_t \\ \Sigma y_{t-1} & \Sigma y_{t-1}^2 & \Sigma y_{t-1} z_t \\ \Sigma z_t & \Sigma z_t y_{t-1} & \Sigma z_t^2 \end{bmatrix}^{-1} \begin{bmatrix} \Sigma y_t \\ \Sigma y_{t-1} y_t \\ \Sigma z_t y_t \end{bmatrix}.$$

In answering the following components of question (3), you do not need to provide proofs of any statements, but will receive full credit for very brief answers such as "yes," "no," or "it cannot be determined from the information given" (though for the latter, briefly state the added information you would need to answer the question).

a.) (5 points) Is $\hat{\gamma}$ an unbiased estimate of γ?

b.) (5 points) Is $\hat{\gamma}$ a consistent estimate of γ?

c.) (10 points) Suppose one proposes to test the null hypothesis $\beta = \beta_0$ by calculating the usual OLS t-statistic and comparing it with Student t critical values. Is this approach basically sound or basically unsound? Why?

d.) (10 points) Suppose one proposes to test the null hypothesis $\beta = \beta_0$ by calculating the t-statistic using autocorrelation- and heteroskedasticity- consistent standard errors with Newey-West lag equal to one, and comparing the resulting statistic with $N(0, 1)$ critical values. Is this approach basically sound or basically unsound? Why?

e.) (10 points) What do you think would be the best way to test the hypothesis $\beta = \beta_0$ for this problem? Your answer could be one of the procedures described in (c) or (d), or some other procedure that you should describe here.

4.) (75 points total) Consider the following regression model:

$$y_t = \alpha + \beta z_t + u_t$$

where $\{u_t\}$ is a martingale difference sequence and $\{y_t, z_t, u_t\}$ are stationary and ergodic. Suppose that

$$\text{plim } T^{-1} \sum_{t=1}^{T} z_t = 0 \qquad \text{plim } T^{-1} \sum_{t=1}^{T} z_t^2 = q > 0 \qquad \text{plim } T^{-1} \sum_{t=1}^{T} z_t u_t = \gamma > 0.$$

a.) (15 points) Give the mathematical definition of a plim and explain in words what each of the above three expressions is assuming about the nature of z_t and its relation to u_t

b.) (20 points) Let $\hat{\alpha}$ and $\hat{\beta}$ be the usual OLS estimates defined by

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} T & \Sigma z_t \\ \Sigma z_t & \Sigma z_t^2 \end{bmatrix}^{-1} \begin{bmatrix} \Sigma y_t \\ \Sigma z_t y_t \end{bmatrix}.$$

Calculate the plim of $\hat{\beta}$. Explain in words the significance of this result.

c.) (15 points) State mathematically the conditions for a vector $\mathbf{x}_t$ to be usable instrumental variables for this problem.

d.) (20 points) Assuming that the vector $\mathbf{x}_t$ does consist of appropriate instruments, give the formula for the two-stage least-squares estimates of α and β.

e.) (5 points) If you were to pose this two-stage least squares estimation problem as a generalized method of moments problem, what function $\mathbf{h}(\mathbf{w}_t, \boldsymbol{\theta})$ would correspond to the orthogonality condition? (Hint: if you've forgotten the notation for GMM, see the next question).

Econ 220B
James Hamilton

5.) (80 points total) If $E[\mathbf{h}(\boldsymbol{\theta}, \mathbf{w}_t)] = \mathbf{0}$, the GMM estimate of $\boldsymbol{\theta}$ is defined as the value that minimizes

$$T[\mathbf{g}(\boldsymbol{\theta}; \mathbf{Y}_T)]'\hat{\mathbf{S}}^{-1}[\mathbf{g}(\boldsymbol{\theta}; \mathbf{Y}_T)]$$

where

$$\mathbf{g}(\boldsymbol{\theta}; \mathbf{Y}_T) = T^{-1}\sum_{t=1}^{T} \mathbf{h}(\boldsymbol{\theta}, \mathbf{w}_t)$$

and $\hat{\mathbf{S}}$ is a consistent estimate of

$$\mathbf{S} = \lim_{T\to\infty} T^{-1} \sum_{t=1}^{T} \sum_{v=-\infty}^{\infty} E\left\{[\mathbf{h}(\boldsymbol{\theta}_0, \mathbf{w}_t)]\left[\mathbf{h}(\boldsymbol{\theta}_0, \mathbf{w}_{t-v})\right]'\right\}.$$

The usual asymptotic distribution of the GMM estimator is

$$\sqrt{T}(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0) \xrightarrow{L} N(\mathbf{0}, \mathbf{V})$$

where $\mathbf{V}$ can be estimated consistently from

$$\hat{\mathbf{V}} = (\hat{\mathbf{D}}\hat{\mathbf{S}}^{-1}\hat{\mathbf{D}}')^{-1}$$

$$\hat{\mathbf{D}}' = \left.\frac{\partial \mathbf{g}(\boldsymbol{\theta}; \mathbf{Y}_T)}{\partial \boldsymbol{\theta}'}\right|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}.$$

Suppose that $y_t \sim$ i.i.d. $N(\mu, \sigma^2)$ so that the density is

$$f(y_t) = (2\pi\sigma^2)^{-1/2} \exp\left[-\frac{(y_t-\mu)^2}{2\sigma^2}\right].$$

a.) (10 points) Calculate the log likelihood for a sample of size T,

$$\mathcal{L}(\boldsymbol{\theta}) = \log[f(y_1, y_2, ..., y_T; \boldsymbol{\theta})]$$

where $\boldsymbol{\theta} = (\mu, \sigma^2)'$.

b.) (10 points) Find the maximum likelihood estimates $\hat{\mu}$ and $\hat{\sigma}^2$ for this example.

c.) (10 points) If one were to view this maximum likelihood estimation as an example of GMM, what vector-valued function corresponds to $\mathbf{h}(\boldsymbol{\theta}, \mathbf{w}_t)$?

d.) (10 points) Prove directly for this problem (not relying on the general results about GMM in a maximum likelihood context) that $E[\mathbf{h}(\boldsymbol{\theta}, \mathbf{w}_t)] = \mathbf{0}$.

Econ 220B
James Hamilton

e.) (10 points) Calculate $\mathbf{S}$ for this example. (Hint: remember that $E(y_t - \mu)^3 = 0$, $E(y_t - \mu)^4 = 3\sigma^4$, and that the scores form a martingale difference sequence.)

f.) (10 points) Calculate the value of $\hat{\mathbf{D}}'$ for this example. (Hint: as a check on your results for (e) and (f), remember that plim $\hat{\mathbf{D}}' = -\mathbf{S}$.)

g.) (10 points) Using the results from (e) and (f) and the expression for $\hat{\mathbf{V}}$ given above, show that we are justified in treating $\hat{\mu}$ as if approximately distributed $N(\mu, \hat{\sigma}^2/T)$.

h.) (10 points) Looking again at the results you obtained to parts (b) and (g), comment on what quasi-maximum likelihood means in this context.