

Econ 220B, Winter 2009
James Hamilton

Problem Set 5
Due Wednesday, Feb 25

This problem set explores what is known as the Anderson-Rubin test, in which you will show that this is a test that can be used to test a null hypothesis in an instrumental-variable setting even when the instruments are irrelevant. The reference for the test is T. W. Anderson and Herman Rubin, *The Annals of Mathematical Statistics*, Vol. 20, No. 1 (Mar., 1949), pp. 46-63.

The equation of interest is

$$y_t = z_t\beta + u_t \tag{1}$$

which does not satisfy the OLS orthogonality condition:

$$E(z_t u_t) \neq 0.$$

However, we do have an $(r \times 1)$ vector of instruments \mathbf{x}_t which satisfy $E(\mathbf{x}_t u_t) = 0$ and the idea is to use these instruments to construct a valid test of the hypothesis $H_0: \beta = \beta_0$. We will assume that the vector $(y_t, z_t, \mathbf{x}_t)'$ is stationary and ergodic. The basic idea will make use of the population linear projection of z_t on \mathbf{x}_t ,

$$z_t = \mathbf{x}_t' \boldsymbol{\delta} + v_t \tag{2}$$

so that $E(\mathbf{x}_t v_t) = \mathbf{0}$ by the definition of $\boldsymbol{\delta}$.

1.) Subtract $z_t\beta_0$ from both sides of (1) and substitute equation (2) into the result to obtain an expression of the form

$$\tilde{y}_t = \mathbf{x}_t' \boldsymbol{\psi} + w_t \tag{3}$$

where $\tilde{y}_t = y_t - z_t\beta_0$. Find the expressions for $\boldsymbol{\psi}$ and w_t , and show that under H_0 , $\boldsymbol{\psi} = \mathbf{0}$ and $E(\mathbf{x}_t w_t) = \mathbf{0}$.

2.) Given your results for question 1, it makes sense to estimate equation (3) by OLS. What would the expressions be for $\hat{\boldsymbol{\psi}}$ and the usual F test of $\boldsymbol{\psi} = \mathbf{0}$?

3.) Suppose that

$$T^{-1} \sum \mathbf{x}_t \mathbf{x}_t' \xrightarrow{p} \mathbf{Q} \text{ for } \mathbf{Q} \text{ a matrix of rank } r \tag{4}$$

$$T^{-1/2} \sum \mathbf{x}_t u_t \xrightarrow{L} N(\mathbf{0}, \sigma^2 \mathbf{Q}). \tag{5}$$

What sort of additional assumptions would imply conditions (4) and (5)?

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- 4.) Show that under the above assumptions, r times the F statistic you proposed in part (2) has an asymptotic $\chi^2(r)$ distribution. Is this derivation valid even when $\boldsymbol{\delta} = \mathbf{0}$?
- 5.) Suppose we further assume that for

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_T \end{bmatrix} \quad \mathbf{X} = \begin{bmatrix} \mathbf{x}'_1 \\ \vdots \\ \mathbf{x}'_T \end{bmatrix}$$

under H_0 it is the case that $\mathbf{u}|\mathbf{X} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_T)$. Show that under this assumption, the F statistic in question 3 has an exact small-sample $F(r, T - r)$ distribution. Does this conclusion hold even if $\boldsymbol{\delta} = \mathbf{0}$?