I. Bayesian econometrics

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Generic Bayesian problem:

$p(\mathbf{Y}|\boldsymbol{\theta})$ = likelihood (known)

$p(\boldsymbol{\theta})$ = prior (known)

goal: calculate

$$p(\boldsymbol{\theta}|\mathbf{Y}) = \frac{p(\mathbf{Y}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{G}$$

for $G = \int p(\mathbf{Y}|\boldsymbol{\theta})p(\boldsymbol{\theta})d\boldsymbol{\theta}$

Analytical approach: choose $p(\boldsymbol{\theta})$ from a family such that G can be found with clever algebra.

Numerical approach: satisfied to be able to generate draws

$$\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}, \ldots, \boldsymbol{\theta}^{(D)}$$

from the distribution $p(\boldsymbol{\theta}|\mathbf{Y})$ without ever knowing the distribution (i.e., without calculating G)

Importance sampling:

Step (1): Generate $\boldsymbol{\theta}^{(j)}$ from an (essentially arbitrary) "importance density" $g(\boldsymbol{\theta})$.

Step (2): Calculate $\omega^{(j)} = \frac{p[\mathbf{Y}|\boldsymbol{\theta}^{(j)}]p[\boldsymbol{\theta}^{(j)}]}{g[\boldsymbol{\theta}^{(j)}]}$.

Step (3): Weight the draw $\boldsymbol{\theta}^{(j)}$ by $\omega^{(j)}$ to simulate distribution of $p(\boldsymbol{\theta}|\mathbf{Y})$.

Examples:

$$E(\boldsymbol{\theta}|\mathbf{Y}) = \int \boldsymbol{\theta} p(\boldsymbol{\theta}|\mathbf{Y}) d\boldsymbol{\theta}$$
$$\simeq \frac{\sum_{j=1}^{D} \boldsymbol{\theta}^{(j)} \omega^{(j)}}{\sum_{j=1}^{D} \omega^{(j)}}$$
$$\equiv \boldsymbol{\theta}^*$$

$$\text{Var}(\boldsymbol{\theta}|\mathbf{Y}) \simeq \frac{\sum_{j=1}^{D} (\boldsymbol{\theta}^{(j)} - \boldsymbol{\theta}^*)(\boldsymbol{\theta}^{(j)} - \boldsymbol{\theta}^*)' \omega^{(j)}}{\sum_{j=1}^{D} \omega^{(j)}}$$

$$\text{Prob}(\theta_2 < 0) \simeq \frac{\sum_{j=1}^{D} \delta_{[\theta_2^{(j)} < 0]} \omega^{(j)}}{\sum_{j=1}^{D} \omega^{(j)}}$$

How does this work?

$$\frac{\sum_{j=1}^{D} \boldsymbol{\theta}^{(j)} \omega^{(j)}}{\sum_{j=1}^{D} \omega^{(j)}} = \frac{D^{-1} \sum_{j=1}^{D} \boldsymbol{\theta}^{(j)} \omega^{(j)}}{D^{-1} \sum_{j=1}^{D} \omega^{(j)}}$$

Numerator:

$$D^{-1} \sum_{j=1}^{D} \boldsymbol{\theta}^{(j)} \omega^{(j)} \xrightarrow{p} E[\boldsymbol{\theta}^{(j)} \omega^{(j)}]$$

$$= \int \boldsymbol{\theta} \omega(\boldsymbol{\theta}) g(\boldsymbol{\theta}) d\boldsymbol{\theta}$$

$$= \int \boldsymbol{\theta} \frac{p(\mathbf{Y}|\boldsymbol{\theta}) p(\boldsymbol{\theta})}{g(\boldsymbol{\theta})} g(\boldsymbol{\theta}) d\boldsymbol{\theta}$$

$$= \int \boldsymbol{\theta} p(\mathbf{Y}|\boldsymbol{\theta}) p(\boldsymbol{\theta}) d\boldsymbol{\theta}$$

Denominator:

$$D^{-1}\sum_{j=1}^{D}\omega^{(j)} \xrightarrow{p} E[\omega^{(j)}]$$
$$= \int \omega(\boldsymbol{\theta})g(\boldsymbol{\theta})d\boldsymbol{\theta}$$
$$= \int \frac{p(\mathbf{Y}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{g(\boldsymbol{\theta})}g(\boldsymbol{\theta})d\boldsymbol{\theta}$$
$$= \int p(\mathbf{Y}|\boldsymbol{\theta})p(\boldsymbol{\theta})d\boldsymbol{\theta}$$
$$= p(\mathbf{Y})$$

Conclusion:

$$\frac{\sum_{j=1}^{D}\boldsymbol{\theta}^{(j)}\omega^{(j)}}{\sum_{j=1}^{D}\omega^{(j)}} \xrightarrow{p} \frac{\int \boldsymbol{\theta}p(\mathbf{Y}|\boldsymbol{\theta})p(\boldsymbol{\theta})d\boldsymbol{\theta}}{p(\mathbf{Y})}$$
$$= \int \boldsymbol{\theta}p(\boldsymbol{\theta}|\mathbf{Y})d\boldsymbol{\theta}$$

What's required of $g(.)$?

$\omega^{(j)} = \frac{p[\mathbf{Y}|\boldsymbol{\theta}^{(j)}]p[\boldsymbol{\theta}^{(j)}]}{g[\boldsymbol{\theta}^{(j)}]}$ should satisfy law of large numbers.

$\omega^{(j)}$ needs finite variance– $g(.)$ must have fatter tails than $p(\mathbf{Y}|\boldsymbol{\theta})p(\boldsymbol{\theta})$.

Caveat:

Always produces an answer, needs to be checked.

(1) Try special cases where result is known analytically.

(2) Try different $g(.)$ to see if get the same result.

(3) Use analytic results for components of θ in order to keep dimension that must be importance-sampled small.

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   2. The Gibbs sampler

Suppose the parameter vector $\boldsymbol{\theta}$ can be partitioned as $\boldsymbol{\theta}' = (\boldsymbol{\theta}_1', \boldsymbol{\theta}_2', \boldsymbol{\theta}_3')$ with the property that $p(\boldsymbol{\theta}|\mathbf{Y})$ is of unknown form but

$p(\boldsymbol{\theta}_1|\mathbf{Y}, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3)$

$p(\boldsymbol{\theta}_2|\mathbf{Y}, \boldsymbol{\theta}_1, \boldsymbol{\theta}_3)$

$p(\boldsymbol{\theta}_3|\mathbf{Y}, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$

are of known form (same idea works for 2, 4, or n blocks)

(1) Start with arbitrary initial guesses $\boldsymbol{\theta}_1^{(j)}, \boldsymbol{\theta}_2^{(j)}, \boldsymbol{\theta}_3^{(j)}$ for $j = 1$.

(2) Generate:

$\boldsymbol{\theta}_1^{(j+1)}$ from $p(\boldsymbol{\theta}_1|\mathbf{Y}, \boldsymbol{\theta}_2^{(j)}, \boldsymbol{\theta}_3^{(j)})$

$\boldsymbol{\theta}_2^{(j+1)}$ from $p(\boldsymbol{\theta}_2|\mathbf{Y}, \boldsymbol{\theta}_1^{(j+1)}, \boldsymbol{\theta}_3^{(j)})$

$\boldsymbol{\theta}_3^{(j+1)}$ from $p(\boldsymbol{\theta}_3|\mathbf{Y}, \boldsymbol{\theta}_1^{(j+1)}, \boldsymbol{\theta}_2^{(j+1)})$

(3) Repeat step (2) for $j = 1, 2, \ldots, D$

Notice the sequence $\{\boldsymbol{\theta}^{(j)}\}_{j=1}^{D}$ is a Markov chain with transition kernel

$$\pi(\boldsymbol{\theta}^{(j+1)}|\boldsymbol{\theta}^{(j)}) = p(\boldsymbol{\theta}_3^{(j+1)}|\mathbf{Y}, \boldsymbol{\theta}_1^{(j+1)}, \boldsymbol{\theta}_2^{(j+1)})$$
$$p(\boldsymbol{\theta}_2^{(j+1)}|\mathbf{Y}, \boldsymbol{\theta}_1^{(j+1)}, \boldsymbol{\theta}_3^{(j)})$$
$$p(\boldsymbol{\theta}_1^{(j+1)}|\mathbf{Y}, \boldsymbol{\theta}_2^{(j)}, \boldsymbol{\theta}_3^{(j)})$$

Under quite general conditions, the realizations from a Markov chain for $D \to \infty$ converge to draws from the ergodic distribution of the chain $\pi(\theta)$ satisfying

$$\pi(\theta^{(j+1)}) = \int_{\Re^k} \pi(\theta^{(j+1)}|\theta^{(j)})\pi(\theta^{(j)})d\theta^{(j)}$$

Claim: the ergodic distribution of this chain corresponds to the posterior distribution:

$$\pi(\theta) = p(\theta|\mathbf{Y})$$

Proof:

$$\int_{\Re^k} \pi(\theta^{(j+1)}|\theta^{(j)})\pi(\theta^{(j)})d\theta^{(j)}$$

$$= \int_{\Re^k} \{p(\theta_3^{(j+1)}|\mathbf{Y}, \theta_1^{(j+1)}, \theta_2^{(j+1)})$$

$$p(\theta_2^{(j+1)}|\mathbf{Y}, \theta_1^{(j+1)}, \theta_3^{(j)})$$

$$p(\theta_1^{(j+1)}|\mathbf{Y}, \theta_2^{(j)}, \theta_3^{(j)})\}$$

$$p(\theta^{(j)}|\mathbf{Y})d\theta^{(j)}$$

$$= \int_{\Re^k} p(\boldsymbol{\theta}^{(j+1)}, \boldsymbol{\theta}^{(j)} | \mathbf{Y}) d\boldsymbol{\theta}^{(j)}$$
$$= p(\boldsymbol{\theta}^{(j+1)} | \mathbf{Y})$$

Implication: if we throw out the first D_0 draws (for D_0 large), then $\boldsymbol{\theta}^{(D_0+1)}, \boldsymbol{\theta}^{(D_0+2)}, \ldots, \boldsymbol{\theta}^{(D)}$ represent draws from the posterior distribution $p(\boldsymbol{\theta}|\mathbf{Y})$.

Checks:

(1) Change $\boldsymbol{\theta}^{(1)} \Rightarrow$ same answer?

(2) Change $D_0, D \Rightarrow$ same answer?

(3) Plot $\boldsymbol{\theta}^{(j)}$ as function of j.

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3. Metropolis-Hastings algorithm

Suppose $\{s_t\}_{t=1}^{T}$ is an ergodic K-state Markov chain,

$$s_t \in \{1,2,\ldots,K\}$$

with transition probabilities

$$p_{ij} = \Pr[s_t = j | s_{t-1} = i]$$

$$\sum_{j=1}^{K} p_{ij} = 1 \quad \text{for } i = 1,\ldots,K$$

$$p_{ij} \geq 0 \quad \text{for } i,j = 1,\ldots,K$$

The ergodic or unconditional probabilities satisfy

$$\Pr[s_t = j] = \sum_{i=1}^{K} \Pr[s_t = j, s_{t-1} = i]$$

$$\pi_j = \sum_{i=1}^{K} p_{ij}\pi_i$$

Proposition: Suppose we can find a set of numbers $f_1, f_2, .., f_K$ such that

$f_j \geq 0$ for $j = 1, \ldots, K$

$\sum_{j=1}^{K} f_j = 1$

$f_i p_{ij} = f_j p_{ji}$

Then $f_j = \pi_j$

Proof: We're given that

$f_i p_{ij} = f_j p_{ji}$

sum over i:

$\sum_{i=1}^{K} f_i p_{ij} = f_j \sum_{i=1}^{K} p_{ji} = f_j$

which satisfy definitions of π_i,

$\sum_{i=1}^{K} \pi_i p_{ij} = \pi_j$

Works also for continuous-valued Markov chains.

If $\mathbf{x}_t \in \Re^k$ is Markov with transition kernel $p(\mathbf{x}, \mathbf{y})$ (meaning that):

$$\Pr[\mathbf{x}_t \in A | \mathbf{x}_{t-1} = \mathbf{x}] = \int_A p(\mathbf{x}, \mathbf{y}) d\mathbf{y}$$

then the ergodic density $\pi(\mathbf{y})$, which signifies that

$$\Pr[\mathbf{x}_t \in A] = \int_A \pi(\mathbf{y}) d\mathbf{y},$$

satisfies

$$\pi(\mathbf{y}) = \int_{\Re^k} p(\mathbf{x}, \mathbf{y}) \pi(\mathbf{x}) d\mathbf{x}$$

Proposition: if

$f(\mathbf{x}) \geq 0$ for all $\mathbf{x}$

$\int_{\Re^k} f(\mathbf{x}) d\mathbf{x} = 1$

$f(\mathbf{x}) p(\mathbf{x}, \mathbf{y}) = f(\mathbf{y}) p(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y}$

then

$\pi(\mathbf{x}) = f(\mathbf{x})$

Goal in Metropolis-Hastings:

We know how to calculate $h\pi(\mathbf{x})$ (where h may be an unknown constant) and want to sample from it.

Solution: generate a sample $\{\mathbf{x}_t\}$ from a Markov chain whose ergodic density is $\pi(\mathbf{x})$

How MH works:

We previously generated $\mathbf{x}_{t-1} = \mathbf{x}$

We now generate a candidate $\mathbf{y}$ from some known density $q(\mathbf{x}, \mathbf{y})$

We'll then set $\mathbf{x}_t = \mathbf{y}$ if $\pi(\mathbf{y})/\pi(\mathbf{x})$ is big and otherwise keep $\mathbf{x}_t = \mathbf{x}$

Let $\alpha(\mathbf{x}, \mathbf{y})$ be probability we set $\mathbf{x}_t = \mathbf{y}$

If $\pi(\mathbf{x})q(\mathbf{x},\mathbf{y}) > 0$, then

$$\alpha(\mathbf{x},\mathbf{y}) = \min\left[\frac{\pi(\mathbf{y})q(\mathbf{y},\mathbf{x})}{\pi(\mathbf{x})q(\mathbf{x},\mathbf{y})}, 1\right]$$

otherwise

$$\alpha(\mathbf{x},\mathbf{y}) = 1$$

When $\mathbf{x} \neq \mathbf{y}$, the transition kernel of this chain is $q(\mathbf{x},\mathbf{y})\alpha(\mathbf{x},\mathbf{y})$. To show that $\pi(\mathbf{y})$ is the ergodic density of this chain, we must show that

$$\pi(\mathbf{x})\alpha(\mathbf{x},\mathbf{y})q(\mathbf{x},\mathbf{y}) = \pi(\mathbf{y})\alpha(\mathbf{y},\mathbf{x})q(\mathbf{y},\mathbf{x})$$

But

$$\begin{aligned}
\pi(\mathbf{x})\alpha(\mathbf{x},\mathbf{y})q(\mathbf{x},\mathbf{y}) &= \pi(\mathbf{x})q(\mathbf{x},\mathbf{y})\min\left[\frac{\pi(\mathbf{y})q(\mathbf{y},\mathbf{x})}{\pi(\mathbf{x})q(\mathbf{x},\mathbf{y})}, 1\right] \\
&= \min[\pi(\mathbf{y})q(\mathbf{y},\mathbf{x}), \pi(\mathbf{x})q(\mathbf{x},\mathbf{y})] \\
&= \pi(\mathbf{y})\alpha(\mathbf{y},\mathbf{x})q(\mathbf{y},\mathbf{x})
\end{aligned}$$

Options for candidate density:

(1) independent $q(\mathbf{y},\mathbf{x}) = q(\mathbf{y})$

e.g., $\mathbf{y} \sim N(\boldsymbol{\lambda},\boldsymbol{\Lambda})$ where $\boldsymbol{\lambda}$ is our guess of mean of $\pi(\mathbf{y})$

Options for candidate density:

(2) random walk

$q(\mathbf{y},\mathbf{x}) = q(\mathbf{y}-\mathbf{x})$

e.g.,

$$q(\mathbf{y},\mathbf{x}) = (2\pi)^{-n/2}|\boldsymbol{\Lambda}|^{-1/2} \times \exp[(-1/2)(\mathbf{y}-\mathbf{x})'\boldsymbol{\Lambda}^{-1}(\mathbf{y}-\mathbf{x})]$$