

II. Vector autoregressions

- A. Introduction
- B. Normal-Wishart priors for VAR's
- C. Bayesian analysis of structural VAR's
- D. Identification using inequality constraints

Structural model:

$$\mathbf{B}_0 \mathbf{y}_t = \mathbf{b}_0 + \mathbf{B}_1 \mathbf{y}_{t-1} + \mathbf{B}_2 \mathbf{y}_{t-2} \\ + \dots + \mathbf{B}_p \mathbf{y}_{t-p} + \mathbf{u}_t$$

$$E(\mathbf{u}_t \mathbf{u}'_{t-m}) = \begin{cases} \mathbf{I}_n & \text{for } m = 0 \\ \mathbf{0} & \text{for } m \neq 0 \end{cases}$$

u_{ft} one standard deviation shock
to Fed policy

P_t GDP price deflator (y_{3t})

N_t nonborrowed reserves (y_{4t})

r_t fed funds rate (y_{6t})

$$\frac{\partial P_{t+s}}{\partial u_{ft}} \leq 0 \text{ for } s = 0, 1, 2, 3, 4, 5$$

$$\frac{\partial N_{t+s}}{\partial u_{ft}} \leq 0 \text{ for } s = 0, 1, 2, 3, 4, 5$$

$$\frac{\partial r_{t+s}}{\partial u_{ft}} \geq 0 \text{ for } s = 0, 1, 2, 3, 4, 5$$

VAR (reduced-form)

$\mathbf{y}_t = (n \times 1)$ vector

$$\mathbf{y}_t = \mathbf{c} + \Phi_1 \mathbf{y}_{t-1} + \Phi_2 \mathbf{y}_{t-2} + \dots + \Phi_p \mathbf{y}_{t-p} + \boldsymbol{\varepsilon}_t$$

$$\boldsymbol{\varepsilon}_t = \mathbf{y}_t - \hat{\mathbf{y}}_{t|t-1}$$

$$E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t') = \boldsymbol{\Omega}$$

$$\boldsymbol{\Psi}_s = \frac{\partial \mathbf{y}_{t+s}}{\partial \boldsymbol{\varepsilon}_t'}$$

$$\boldsymbol{\Omega} = \mathbf{P}\mathbf{P}' \text{ (}\mathbf{P} \text{ lower triangular)}$$

$$\mathbf{q}_s = \frac{\partial \mathbf{y}_{t+s}}{\partial u_{ft}}$$

Claim: there exists a unique $\boldsymbol{\alpha}$ such that

$$\mathbf{q}_s = \boldsymbol{\Psi}_s \mathbf{P} \boldsymbol{\alpha}$$

$$\boldsymbol{\alpha}' \boldsymbol{\alpha} = 1$$

In order to satisfy sign constraint,

$$\frac{\partial P_{t+s}}{\partial u_{ft}} \leq 0 \text{ for } s = 0, 1, 2, 3, 4, 5,$$

we must have

$$\mathbf{e}'_3 \Psi_s \mathbf{P} \alpha \leq 0 \text{ for } s = 0, 1, 2, 3, 4, 5$$

Constraints on signs for nonborrowed reserves and fed funds rate require

$$\mathbf{e}'_4 \Psi_s \mathbf{P} \alpha \leq 0 \text{ for } s = 0, 1, 2, 3, 4, 5$$

$$\mathbf{e}'_6 \Psi_s \mathbf{P} \alpha \geq 0 \text{ for } s = 0, 1, 2, 3, 4, 5$$

Estimation strategy (Bayesian approach)
Use noninformative Normal-Wishart priors

(1) Estimate i th equation of VAR by OLS

for $i = 1, 2, \dots, n$

$$\mathbf{x}'_t = (1, \mathbf{y}'_{t-1}, \mathbf{y}'_{t-2}, \dots, \mathbf{y}'_{t-p})$$

$$\hat{\gamma}_i = \left(\sum_{t=1}^T \mathbf{x}_t \mathbf{x}'_t \right)^{-1} \left(\sum_{t=1}^T \mathbf{x}_t y_{it} \right)$$

$$\mathbf{M}^* = \left(\sum_{t=1}^T \mathbf{x}_t \mathbf{x}'_t \right)^{-1}$$

$$\hat{\gamma} = (\hat{\gamma}'_1, \hat{\gamma}'_2, \dots, \hat{\gamma}'_n)'$$

(2) Calculate residual of i th equation and sum of outer products of residuals:

$$\hat{\varepsilon}_{it} = y_{it} - \hat{\gamma}'_i \mathbf{x}_t$$

$$\hat{\boldsymbol{\varepsilon}}_t = (\hat{\varepsilon}_{1t}, \hat{\varepsilon}_{2t}, \dots, \hat{\varepsilon}_{nt})'$$

$$\hat{\mathbf{S}} = \sum_{t=1}^T \hat{\boldsymbol{\varepsilon}}_t \hat{\boldsymbol{\varepsilon}}_t'$$

(3) For $m = 1, 2, \dots, 10,000$, generate an artificial sample of size T of

$(n \times 1)$ vectors $\mathbf{z}_i^{(m)}$ where $\mathbf{z}_i^{(m)} \sim N(\mathbf{0}, \hat{\mathbf{S}}^{-1})$

and calculate the $(n \times n)$ matrix

$$\mathbf{W}^{(m)} = \sum_{i=1}^T \mathbf{z}_i^{(m)} \mathbf{z}_i^{(m)'}$$

(4) Set $\boldsymbol{\Omega}^{(m)} = [\mathbf{W}^{(m)}]^{-1}$

(5) Generate a draw for $\boldsymbol{\gamma}^{(m)}$ from a $N(\hat{\boldsymbol{\gamma}}, \boldsymbol{\Omega}^{(m)} \otimes \mathbf{M}^*)$ distribution

(6) Generate $\tilde{\boldsymbol{\alpha}}^{(m)} \sim N(\mathbf{0}, \mathbf{I}_n)$

$$\boldsymbol{\alpha}^{*(m)} = \tilde{\boldsymbol{\alpha}}^{(m)} / [\tilde{\boldsymbol{\alpha}}^{(m)' \tilde{\boldsymbol{\alpha}}^{(m)}]^{1/2}$$

(7) Flip signs if initial shocks have wrong sign:

$$\begin{bmatrix} \alpha_1^{(m)} \\ \alpha_2^{(m)} \\ \alpha_3^{(m)} \\ \alpha_4^{(m)} \\ \alpha_5^{(m)} \\ \alpha_6^{(m)} \end{bmatrix} = \begin{bmatrix} \alpha_1^{*(m)} \\ \alpha_2^{*(m)} \\ \alpha_3^{*(m)} \text{sign}[-\alpha_3^{*(m)}] \\ \alpha_4^{*(m)} \text{sign}[-\alpha_4^{*(m)}] \\ \alpha_5^{*(m)} \\ \alpha_6^{*(m)} \text{sign}[\alpha_6^{*(m)}] \end{bmatrix}$$

(8) Find Cholesky factorization

$$\Omega^{(m)} = \mathbf{P}^{(m)} \mathbf{P}^{(m)'}$$

(9) Calculate nonorthogonal impulse-response function $\Psi_s^{(m)}$ $s = 0, 1, 2, \dots$

(10) For each draw m , throw out α and draw another if it fails sign restrictions:

$$\mathbf{e}_3' \Psi_s^{(m)} \mathbf{P}^{(m)} \alpha \leq 0$$

$$\mathbf{e}_4' \Psi_s^{(m)} \mathbf{P}^{(m)} \alpha \leq 0$$

$$\mathbf{e}_6' \Psi_s^{(m)} \mathbf{P}^{(m)} \alpha \geq 0$$

for $s = 0, 1, \dots, K$ (K is your choice)

(11) For the retained draws,

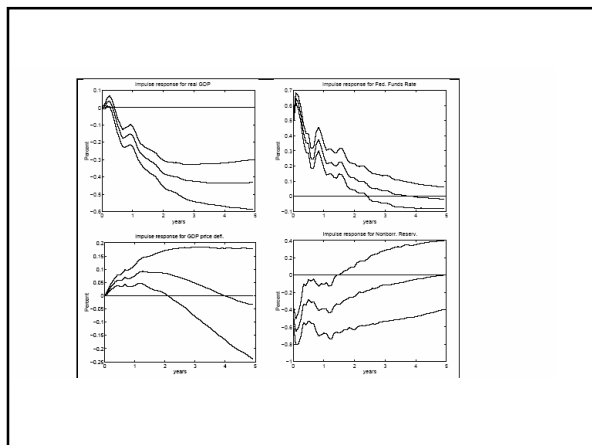
$$\left\{ \Psi_s^{(m)} \mathbf{P}^{(m)} \boldsymbol{\alpha}^{(m)} \right\}_{m=1}^{10,000}$$

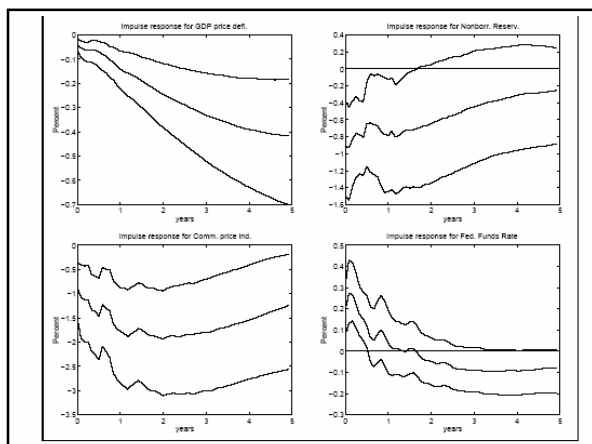
gives the sampling distribution of

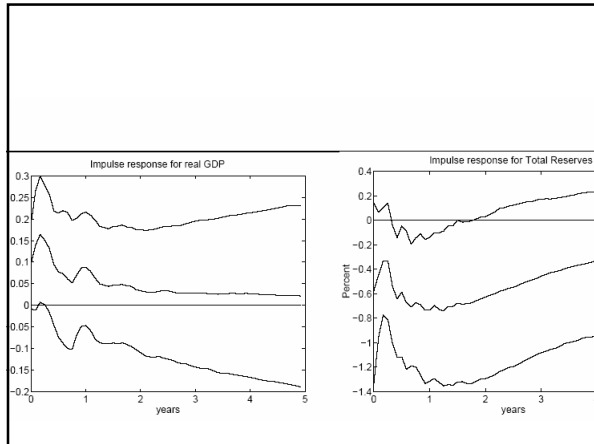
$$\frac{\partial \mathbf{y}_{t+s}}{\partial u_{ft}}$$

e.g., Bayesian posterior estimate is

$$\widehat{\frac{\partial \mathbf{y}_{t+s}}{\partial u_{ft}}} = M^{-1} \sum_{m=1}^M \Psi_s^{(m)} \mathbf{P}^{(m)} \boldsymbol{\alpha}^{(m)}$$







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Del Negro and Schorfheide:

$X_t(j)$ = output of firm j
 X_t = output of representative firm in symmetric equilibrium
 $h_t(j)$ = labor hired by firm j
 A_t = productivity
 $X_t(j) = A_t h_t(j)$

C_t = consumption of representative household
 G_t = government spending
 $X_t = C_t + G_t$
 $G_t = \zeta_t X_t$

Sources of randomness:

(1) technology shock

$$\log A_t = \log \gamma + \log A_{t-1} + \tilde{z}_t$$

$$\tilde{z}_t = \rho_z \tilde{z}_{t-1} + u_{zt}$$

(2) government spending shock

$$\tilde{g}_t = g^* - \log(1 - \zeta_t)$$

$$\tilde{g}_t = \rho_g \tilde{g}_{t-1} + u_{gt}$$

(3) Fed policy shock (Taylor rule)

$$\tilde{r}_t = \rho_R \tilde{r}_{t-1} + (1 - \rho_R)(\psi_1 \tilde{\pi}_t + \psi_2 \tilde{x}_t) + u_{ft}$$

Will use these three shocks to explain output growth, fed funds rate, and inflation rate

$$U(C_t, M_t/P_t, h_t) = \frac{(C_t/A_t)^{1-\tau}}{1-\tau} + \chi \log(M_t/P_t) - h_t$$

Euler equation:

$$U_C(C_t, M_t/P_t, h_t) = \beta E_t \{ U_C(C_{t+1}, M_{t+1}/P_{t+1}, h_{t+1}) \times (1 + i_t)/(1 + \pi_{t+1}) \}$$

Substitute in equilibrium condition

$$C_t = (1 - \zeta_t) X_t$$

and log linearize:

$$\tilde{x}_t = E_t \tilde{x}_{t+1} - \tau^{-1} [\tilde{r}_t - E_t \tilde{\pi}_{t+1}] + (1 - \rho_G) \tilde{g}_t + (\rho_z/\tau) \tilde{z}_t$$

Euler equation:

$$\tilde{x}_t = E_t \tilde{x}_{t+1} - \tau^{-1} [\tilde{r}_t - E_t \tilde{\pi}_{t+1}] + (1 - \rho_G) \tilde{g}_t + (\rho_z / \tau) \tilde{z}_t$$

Firms face quadratic costs in adjusting prices:

$$\tilde{\pi}_t = \beta E_t \tilde{\pi}_{t+1} + \kappa (\tilde{x}_t - \tilde{g}_t)$$

- θ vector of structural parameters
- γ average productivity growth
- β personal discount rate
- κ slope of Phillips curve (function of adjustment costs and demand elasticity)

- τ coefficient of relative risk aversion
- ψ_1, ψ_2 coefficients on inflation and output in Taylor rule
- ρ_g, ρ_z serial correlation of govt. spending and productivity shocks
- $\sigma_f, \sigma_g, \sigma_z$ standard deviations of structural shocks

Observable data:

y_{1t} quarterly real GDP growth

y_{2t} quarterly CPI inflation

y_{3t} average fed funds rate over quarter

$$y_{1t} = \log(\gamma) + \Delta\tilde{x}_t + \tilde{z}_t$$

$$y_{2t} = \log(\pi^*) + \tilde{\pi}_t$$

$$y_{3t} = \log(\beta/\gamma) + \log(\pi^*) + \tilde{r}_t$$

Given θ , can use numerical methods to calculate model's predictions for

$$E(\mathbf{y}_t' \mathbf{y}_t') = \mathbf{C}(\theta)$$

$$E(\mathbf{x}_t' \mathbf{x}_t') = \mathbf{A}(\theta)$$

$$\mathbf{x}_t = (1, \mathbf{y}_{t-1}', \mathbf{y}_{t-2}', \dots, \mathbf{y}_{t-p}')'$$

$$E(\mathbf{x}_t' \mathbf{y}_t') = \mathbf{B}(\theta)$$

prior for Ω :

$$\Omega^{-1} | \theta \sim W(\tilde{T}, \Lambda(\theta))$$

\tilde{T} number of observations we want
to treat theoretical model as
equivalent to

$$\Lambda(\theta) = \tilde{T} [\mathbf{C}(\theta) - \mathbf{B}(\theta)' \mathbf{A}(\theta)^{-1} \mathbf{B}(\theta)]$$

prior for γ :

$$\gamma | \Omega, \theta \sim N(\mathbf{m}(\theta), \Omega \otimes \mathbf{M}(\theta))$$

$$\mathbf{m}(\theta) = \text{vec}[\mathbf{A}(\theta)^{-1} \mathbf{B}(\theta)]$$

$$\mathbf{M}(\theta) = [\tilde{T} \mathbf{A}(\theta)]^{-1}$$

prior for θ :

τ Gamma with mean 2
standard deviation 0.5

ψ_1 Gamma with mean 1.5
standard deviation 0.25

prior for all parameters:

$$f(\theta)f(\Omega^{-1}|\theta)f(\gamma|\Omega, \theta)$$

Have analytical expressions for

$f(\mathbf{Y}|\gamma, \Omega)$ from likelihood

(same as $f(\mathbf{Y}|\gamma, \Omega, \theta)$)

$f(\gamma, \Omega|\theta)$ from prior

$f(\gamma, \Omega|\theta, \mathbf{Y})$ from posterior

From these we could calculate
the value of

$$f(\mathbf{Y}|\theta) = \frac{f(\mathbf{Y}|\gamma, \Omega, \theta)f(\gamma, \Omega|\theta)}{f(\gamma, \Omega|\mathbf{Y}, \theta)}$$

Using this and prior $f(\theta)$, we can use numerical methods (Metropolis-Hastings or importance sampling) to sample from

$$f(\theta|\mathbf{Y}) \propto f(\mathbf{Y}|\theta)f(\theta)$$

to get inference about true structural parameters

90% posterior confidence

$$\tau \in (1.3, 2.8)$$

$$\psi_1 \in (1.0, 1.6)$$

By sampling $\theta^{(m)}$ from $f(\theta|\mathbf{Y})$ and then $\Omega^{(m)}$ from $f(\Omega|\theta^{(m)}, \mathbf{Y})$ and $\gamma^{(m)}$ from $f(\gamma|\Omega^{(m)}, \theta^{(m)}, \mathbf{Y})$, we can generate posterior distribution of VAR parameters

Identifying impulse-response
function using structural model

$$\begin{bmatrix} u_{gt} \\ u_{zt} \\ u_{ft} \end{bmatrix} = \begin{bmatrix} \sigma_g v_{gt} \\ \sigma_z v_{zt} \\ \sigma_f v_{ft} \end{bmatrix}$$

$$E(\mathbf{v}_t \mathbf{v}_t') = \mathbf{I}_3$$

$$\mathbf{A}_0 \mathbf{y}_t = \mathbf{a} + \mathbf{A}_1 \mathbf{y}_{t-1} + \mathbf{A}_2 \mathbf{y}_{t-2} + \dots + \mathbf{A}_p \mathbf{y}_{t-p} + \mathbf{v}_t$$

Can calculate from structural
model

$$\mathbf{A}_0(\boldsymbol{\theta})^{-1} = \frac{\partial \mathbf{y}_t}{\partial \mathbf{v}_t'}$$

One option:

$$\frac{\partial \mathbf{y}_{t+s}}{\partial \mathbf{v}_t'} = \frac{\partial \mathbf{y}_{t+s}}{\partial \boldsymbol{\varepsilon}_t'} \frac{\partial \boldsymbol{\varepsilon}_t}{\partial \mathbf{v}_t'}$$

$$= \boldsymbol{\Psi}_s^{(m)} \mathbf{A}_0(\boldsymbol{\theta}^{(m)})^{-1}$$

Problem:

$\mathbf{\Omega}^{(m)} \neq \mathbf{A}_0(\boldsymbol{\theta}^{(m)})^{-1} [\mathbf{A}_0(\boldsymbol{\theta}^{(m)})^{-1}]'$
 $\Psi^{(m)} \mathbf{A}_0(\boldsymbol{\theta}^{(m)})^{-1}$ OK for impulse-
response, but variance shares
won't add up

Another option: QR factorization

Proposition: Any $(n \times k)$ matrix \mathbf{C}
can be written as

$$\mathbf{C} = \mathbf{Q} \mathbf{R}$$

$(n \times k)$ $(n \times n)(n \times k)$

where $\mathbf{Q}'\mathbf{Q} = \mathbf{I}_n$

and \mathbf{R} is upper triangular

How to find: qr command
in Gauss or Matlab

(1) Find QR factorization of

$$[\mathbf{A}_0(\boldsymbol{\theta}^{(m)})^{-1}]' = \mathbf{Q}(\boldsymbol{\theta}^{(m)}) \mathbf{R}(\boldsymbol{\theta}^{(m)})$$

$$\mathbf{A}_0(\boldsymbol{\theta}^{(m)})^{-1} = [\mathbf{R}(\boldsymbol{\theta}^{(m)})]' [\mathbf{Q}(\boldsymbol{\theta}^{(m)})]'$$

$[\mathbf{R}(\boldsymbol{\theta}^{(m)})]'$ is thus lower triangular

(2) Find Cholesky factorization of

$$\mathbf{\Omega}^{(m)} = \mathbf{P}^{(m)} [\mathbf{P}^{(m)}]'$$

so $\mathbf{P}^{(m)}$ is lower triangular

Assumption: $\mathbf{P}^{(m)} \simeq [\mathbf{R}(\boldsymbol{\theta}^{(m)})]'$

(3) Use

$$\boldsymbol{\varepsilon}_t = \mathbf{P}^{(m)} [\mathbf{Q}(\boldsymbol{\theta}^{(m)})]' \mathbf{v}_t$$

$$E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t')$$

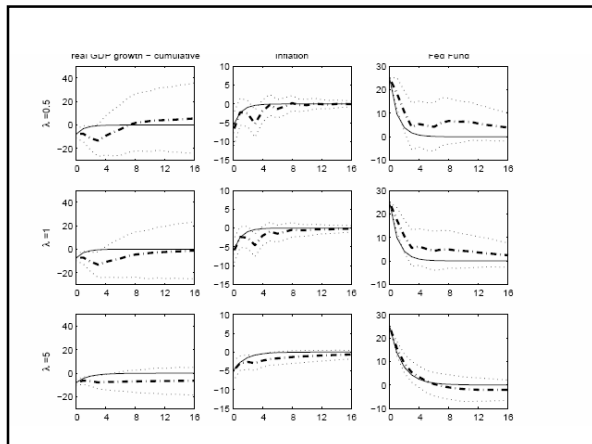
$$= \mathbf{P}^{(m)} [\mathbf{Q}(\boldsymbol{\theta}^{(m)})]' \mathbf{Q}(\boldsymbol{\theta}^{(m)}) \mathbf{P}^{(m)}$$

$$= \mathbf{P}^{(m)} \mathbf{I}_n [\mathbf{P}^{(m)}]'$$

$$= \mathbf{\Omega}^{(m)}$$

In other words,

$$\frac{\partial \mathbf{y}_{t+s}}{\partial \mathbf{v}_t'} = \boldsymbol{\Psi}_s^{(m)} \mathbf{P}^{(m)} [\mathbf{Q}(\boldsymbol{\theta}^{(m)})]'$$



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F. Selecting priors for DSGEs

Traditional approach:

- set priors for each parameter independently
- just augment with another independent prior when a new parameter is introduced

Model 1:

$$y_t = c + \varepsilon_t$$

$$\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$$

Prior 1:

$$c \sim N(m, \lambda^2)$$

implies:

$$y_t \sim N(m, \sigma_\varepsilon^2 + \lambda^2)$$

Model 2:

$$y_t = c + by_{t-1} + \varepsilon_t$$

Suppose we keep specification of ε_t and prior for c the same, and just add independent prior for b :

$$b \sim U[0, \xi] \quad 0 < \xi < 1$$

Implies:

$$y_t|b \sim N\left(\frac{m}{1-b}, \frac{\sigma_\varepsilon^2 + \lambda^2}{(1-b^2)}\right)$$

$$E(y_t) = E_b[E(y_t|b)] > m$$

$$\text{Var}(y_t) > \sigma_\varepsilon^2 + \lambda^2$$

Better approach: for Model 2
use the prior

$$c|b \sim N(m(1-b), \lambda^2(1-b^2))$$

Separate DSGE parameters
into 3 groups:

(1) steady-state parameters

α = production function parameter

$$Y_t(i) = A_t K_t(i)^\alpha L_t(i)^{1-\alpha}$$

λ_f = mark-up parameter

$$Y_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-(1+\lambda_f)/\lambda_f} Y_t$$

$$\text{labor share} = \frac{1-\alpha}{1+\lambda_f}$$

Del Negro and Schorfheide:

Represent prior as fictitious
measurement:

$$\frac{1-\alpha}{1+\lambda_f} = 0.57 + v$$

$$v \sim N(0, 0.02)$$

Can also supplement with much less informative, separate priors:

$$\alpha \sim U[0, 1]$$

$$\lambda_f \sim \Gamma(0.15, 0.10)$$

Other steady-state parameters include:

β (discount rate)

δ (depreciation rate)

γ (technological growth rate)

π^* (Fed's inflation target)

g^* (government share)

Fictitious observations:

labor share

government share

investment-capital ratio

capital-output ratio

\mathbf{s} = fictitious observations

\mathbf{s} = fictitious observations
 $\mathcal{L}(\mathbf{s}|\theta_{ss})$ = implied likelihood
 $p(\theta_{ss})$ = uninformative prior
 $\hat{p}(\theta_{ss}) \propto \mathcal{L}(\mathbf{s}|\theta_{ss})p(\theta_{ss})$
is proposed prior on θ_{ss}

Separate DSGE parameters
into 3 groups:
(1) steady-state parameters
(2) endogenous parameters

$a_t = \log A_t$
 $a_t = (1 - \rho_a)\gamma + \rho_a a_{t-1} + \sigma_a \varepsilon_{at}$
Usual approach:
pick ρ_a and σ_a to match
autocorrelation and variance
of output

Let \mathbf{y}_t = vector of covariance-stationary variables for which DSGE has implications

Γ_{yy}^* = prior belief for $E(\mathbf{y}_t \mathbf{y}_t')$

$\Gamma_{yy}(\theta)$ = model implications

$\mathbf{x}_t = (1, \mathbf{y}_{t-1}', \mathbf{y}_{t-2}', \dots, \mathbf{y}_{t-p}')'$

Γ_{xy}^* = prior belief for $E(\mathbf{x}_t \mathbf{y}_t')$

$\Phi(\theta)$ = model implications for VAR coefficients

$\Phi(\theta) = \Gamma_{xx}(\theta)^{-1} \Gamma_{xy}(\theta)$

$\Omega(\theta)$ = model implications for VAR residual matrix

$\Omega(\theta) = \Gamma_{yy}(\theta) - \Gamma_{yx}(\theta) \Gamma_{xx}^{-1}(\theta) \Gamma_{xy}(\theta)$

T^* = weight on prior (number of equivalent observations)

$p(\theta_{\text{endo}}) \propto |\Omega(\theta)|^{-(T^*+n+1)/2} \times$

$\exp\left\{-\frac{T^*}{2} \text{tr}[\Omega(\theta)^{-1} \times$

$(\Gamma_{yy}^* - 2\Phi(\theta)\Gamma_{xy}^* + \Phi(\theta)'\Gamma_{xx}^*\Phi(\theta))\right\}$

Separate DSGE parameters into 3 groups:
(1) steady-state parameters
(2) endogenous parameters
(3) exogenous parameters

ζ_p = fraction of firms who are not able to optimize their price
 θ_p = fraction of firms who mechanically adjust price

Taylor Rule parameters
 (ρ_R, ψ_1, ψ_2) followed by Fed to set interest rate R_t :
$$R_t = \rho_R R_{t-1} + (1 - \rho_R)(\psi_1 \pi_t + \psi_2 y_t) + \sigma_R \varepsilon_{Rt}$$

Del Negro and Schorfheide use standard priors for this third set.

Examine implications of how you form the prior for posterior inference.

Posterior distribution for wage inflexibility fraction ζ_w :

usual priors:

mean: 0.52

90% confidence: [0.16, 0.81]

DS priors:

mean: 0.19

90% confidence: [0.10, 0.29]
