III. Linear state-space models

A. State-space representation of a dynamic system

Consider following model

State equation:

$$\underset{r\times 1}{\xi_{t+1}} = \underset{r\times r}{\mathbf{F}}\,\underset{r\times 1}{\xi_t} + \underset{r\times 1}{\mathbf{v}_{t+1}}$$

Observation equation:

$$\underset{n\times 1}{\mathbf{y}_t} = \underset{n\times k}{\mathbf{A}'}\,\underset{k\times 1}{\mathbf{x}_t} + \underset{n\times r}{\mathbf{H}'}\,\underset{r\times 1}{\xi_t} + \underset{n\times 1}{\mathbf{w}_t}$$

Observed variables: $\mathbf{y}_t, \mathbf{x}_t$

Unobserved variables: $\xi_t, \mathbf{v}_t, \mathbf{w}_t$

Matrices of parameters: $\mathbf{F}, \mathbf{A}, \mathbf{H}$

$$\begin{bmatrix} \mathbf{v}_t \\ \mathbf{w}_t \end{bmatrix} \sim \text{i.i.d. } N\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \right)$$

$\mathbf{Q} = r \times r$

$\mathbf{R} = n \times n$

Example 1:

$$\xi_{t+1} = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{r-1} & \phi_r \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \xi_t + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\xi_{j,t+1} = L^{j-1}\xi_{1t}$ for $j = 2, 3, \ldots, r$

$$\xi_{1,t+1} = \phi_1\xi_{1t} + \phi_2 L^1 \xi_{1t} + \phi_3 L^2 \xi_{1t} + \cdots + \phi_p L^{p-1}\xi_{1t} + \varepsilon_{t+1}$$

$$\phi(L)\xi_{1,t+1} = \varepsilon_{t+1}$$

Observation equation:

$$y_t = \mu + \begin{bmatrix} 1 & \theta_1 & \theta_2 & \cdots & \theta_{r-1} \end{bmatrix} \xi_t$$

$$y_t - \mu = \theta(L)\xi_{1t}$$

put together with state equation:

$$\phi(L)\xi_{1t} = \varepsilon_t$$

$$\phi(L)(y_t - \mu) = \theta(L)\varepsilon_t$$

Conclusion: any ARMA process can be written as a state-space model.

Example 2:

C_t = state of business cycle

χ_{it} = idiosyncratic component for sector i

C_t, χ_{it} unobserved

y_{it} = growth in sector i (observed)

$$\boldsymbol{\xi}_t = (C_t, \chi_{1t}, \chi_{2t}, \ldots, \chi_{nt})'$$

$$\boldsymbol{\xi}_{t+1} = \mathbf{F}\boldsymbol{\xi}_t + \mathbf{v}_{t+1}$$

$$\mathbf{F} = \begin{bmatrix} \phi_C & 0 & 0 & \cdots & 0 \\ 0 & \phi_1 & 0 & \cdots & 0 \\ 0 & 0 & \phi_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \phi_r \end{bmatrix}$$

Observation equation:

$$\begin{bmatrix} y_{1t} \\ y_{2t} \\ \vdots \\ y_{nt} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix} + \begin{bmatrix} \gamma_1 & 1 & 0 & \cdots & 0 \\ \gamma_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \gamma_n & 0 & 0 & \cdots & 1 \end{bmatrix} \boldsymbol{\xi}_t$$

Purpose of state-space representation: state vector ξ_t contains all information about system dynamics and forecasting.

$$\xi_{t+1} = \mathbf{F}\xi_t + \mathbf{v}_{t+1}$$

$$\mathbf{y}_t = \mathbf{A}'\mathbf{x}_t + \mathbf{H}'\xi_t + \mathbf{w}_t$$

$$E(\mathbf{y}_{t+j}|\xi_t, \xi_{t-1}, \ldots, \xi_1, \mathbf{y}_t, \mathbf{y}_{t-1}, \ldots, \mathbf{y}_1, \mathbf{x}_{t+j}, \mathbf{x}_{t+j-1}, \ldots, \mathbf{x}_1) = \mathbf{A}'\mathbf{x}_{t+j} + \mathbf{H}'\mathbf{F}^j\xi_t$$

III. Linear state-space models

A. State-space representation of a dynamic system
B. Kalman filter

Purpose of Kalman filter: calculate distribution of ξ_t conditional on

$$\Omega_t = \{\mathbf{y}_t, \mathbf{y}_{t-1}, \ldots, \mathbf{y}_1, \mathbf{x}_t, \mathbf{x}_{t-1}, \ldots, \mathbf{x}_1\}$$

$$\xi_t|\Omega_t \sim N(\widehat{\xi}_{t|t}, \mathbf{P}_{t|t})$$

$$\xi_{t+1} = \mathbf{F}\xi_t + \mathbf{v}_{t+1}$$
$$\mathbf{y}_t = \mathbf{A}'\mathbf{x}_t + \mathbf{H}'\xi_t + \mathbf{w}_t$$
$$\begin{bmatrix} \mathbf{v}_t \\ \mathbf{w}_t \end{bmatrix} \sim \text{i.i.d. } N\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}\right)$$

Begin with the prior:

$\xi_0 \sim N(\hat{\xi}_{0|0}, \mathbf{P}_{0|0})$

$\hat{\xi}_{0|0}$ = prior best guess as to value of ξ_0

$\mathbf{P}_{0|0}$ = uncertainty about this guess

(much uncertainty = large diagonal elements of $\mathbf{P}_{0|0}$)

$$\xi_1 = \mathbf{F}\xi_0 + \mathbf{v}_1$$
$$\xi_1 \sim N(\hat{\xi}_{1|0}, \mathbf{P}_{1|0})$$
$$\hat{\xi}_{1|0} = \mathbf{F}\hat{\xi}_{0|0}$$
$$\mathbf{P}_{1|0} = \mathbf{F}\mathbf{P}_{0|0}\mathbf{F}' + \mathbf{Q}$$

Useful result: suppose that

$$\begin{bmatrix} \mathbf{y}_1|\mathbf{x} \\ \mathbf{y}_2|\mathbf{x} \end{bmatrix} \sim N\left(\begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}\right)$$

where $\boldsymbol{\mu}_i$ and $\boldsymbol{\Sigma}_{ij}$ may depend on $\mathbf{x}$. Then

$$\mathbf{y}_2|\mathbf{y}_1,\mathbf{x} \sim N(\mathbf{m}^*,\mathbf{M}^*)$$

$$\mathbf{m}^* = \boldsymbol{\mu}_2 + \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\mathbf{y}_1 - \boldsymbol{\mu}_1)$$

$$\mathbf{M}^* = \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}$$

Here

$$\begin{bmatrix} \mathbf{y}_1|\mathbf{x}_1,\Omega_0 \\ \boldsymbol{\xi}_1|\mathbf{x}_1,\Omega_0 \end{bmatrix} \sim$$

$$N\left(\begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}\right)$$

$$\boldsymbol{\mu}_2 = \hat{\boldsymbol{\xi}}_{1|0} \qquad \boldsymbol{\Sigma}_{22} = \mathbf{P}_{1|0}$$

$$\boldsymbol{\mu}_1 = \mathbf{A}'\mathbf{x}_1 + \mathbf{H}'\hat{\boldsymbol{\xi}}_{1|0} \qquad \boldsymbol{\Sigma}_{11} = \mathbf{H}'\mathbf{P}_{1|0}\mathbf{H} + \mathbf{R}$$

$$\boldsymbol{\Sigma}_{21} = \mathbf{P}_{1|0}\mathbf{H}$$

Hence

$$\boldsymbol{\xi}_1|\mathbf{y}_1,\mathbf{x}_1,\Omega_0 = \boldsymbol{\xi}_1|\Omega_1 \sim N(\hat{\boldsymbol{\xi}}_{1|1},\mathbf{P}_{1|1})$$

$$\hat{\boldsymbol{\xi}}_{1|1} = \hat{\boldsymbol{\xi}}_{1|0} + \mathbf{P}_{1|0}\mathbf{H}(\mathbf{H}'\mathbf{P}_{1|0}\mathbf{H} + \mathbf{R})^{-1} \times \left(\mathbf{y}_1 - \mathbf{A}'\mathbf{x}_1 - \mathbf{H}'\hat{\boldsymbol{\xi}}_{1|0}\right)$$

$$\mathbf{P}_{1|1} = \mathbf{P}_{1|0} - \mathbf{P}_{1|0}\mathbf{H}(\mathbf{H}'\mathbf{P}_{1|0}\mathbf{H} + \mathbf{R})^{-1}\mathbf{H}'\mathbf{P}_{1|0}$$

Identical calculations: if $\xi_t|\Omega_t \sim N(\xi_{t|t}, \mathbf{P}_{t|t})$,

then $\xi_{t+1}|\Omega_{t+1} \sim N(\hat{\xi}_{t+1|t+1}, \mathbf{P}_{t+1|t+1})$

$$\mathbf{P}_{t+1|t} = \mathbf{F}\mathbf{P}_{t|t}\mathbf{F}' + \mathbf{Q}$$

$$\mathbf{P}_{t+1|t+1} = \mathbf{P}_{t+1|t} - \mathbf{P}_{t+1|t}\mathbf{H}(\mathbf{H}'\mathbf{P}_{t+1|t}\mathbf{H} + \mathbf{R})^{-1}\mathbf{H}'\mathbf{P}_{t+1|t}$$

$$\hat{\xi}_{t+1|t} = \mathbf{F}\hat{\xi}_{t|t}$$

$$\hat{\boldsymbol{\varepsilon}}_{t+1|t} = \mathbf{y}_{t+1} - \mathbf{A}'\mathbf{x}_{t+1} - \mathbf{H}'\hat{\xi}_{t+1|t}$$

$$\hat{\xi}_{t+1|t+1} = \hat{\xi}_{t+1|t} + \mathbf{P}_{t+1|t}\mathbf{H}(\mathbf{H}'\mathbf{P}_{t+1|t}\mathbf{H} + \mathbf{R})^{-1}\hat{\boldsymbol{\varepsilon}}_{t+1|t}$$

Iterating on these calculations for $t = 1, 2, \ldots, T$ to produce the sequences $\{\mathbf{P}_{t|t}\}_{t=1}^{T}$ and $\left\{\hat{\xi}_{t|t}\right\}_{t=1}^{T}$ is called the Kalman filter.

$\hat{\xi}_{t|t}$ is the posterior Bayesian expectation of ξ_t given observation of $\Omega_t = \{\mathbf{y}_t, \mathbf{y}_{t-1}, \ldots, \mathbf{y}_1, \mathbf{x}_t, \mathbf{x}_{t-1}, \ldots, \mathbf{x}_1\}$.

$$\mathbf{P}_{t|t} = E\left(\hat{\xi}_{t|t} - \xi_t\right)\left(\hat{\xi}_{t|t} - \xi_t\right)'$$

where these expectations condition on the values of $\mathbf{F}$, $\mathbf{Q}$, $\mathbf{A}$, $\mathbf{H}$, $\mathbf{R}$.

III. Linear state-space models

A. State-space representation of a dynamic system
B. Kalman filter
C. Using the Kalman filter
   1. Estimating the unknown parameters

Classical perspective

$$\boldsymbol{\xi}_{t+1} = \mathbf{F}\boldsymbol{\xi}_t + \mathbf{v}_{t+1}$$

If eigenvalues of $\mathbf{F}$ are all inside unit circle, set

$$\hat{\boldsymbol{\xi}}_{0|0} = E(\boldsymbol{\xi}_0) = \mathbf{0}$$

$$\mathbf{P}_{0|0} = E(\boldsymbol{\xi}_0\boldsymbol{\xi}_0')$$

$$\mathrm{vec}(\mathbf{P}_{0|0}) = [\mathbf{I}_{r^2} - (\mathbf{F} \otimes \mathbf{F})]^{-1}\,\mathrm{vec}(\mathbf{Q})$$

Let $\boldsymbol{\theta}$ be vector containing unknown elements of $\mathbf{F}, \mathbf{Q}, \mathbf{A}, \mathbf{H}, \mathbf{R}$

$$\mathbf{y}_t|\Omega_{t-1}, \mathbf{x}_t; \boldsymbol{\theta} \sim N(\hat{\mathbf{y}}_{t|t-1}, \mathbf{C}_{t|t-1})$$

$$\hat{\mathbf{y}}_{t|t-1} = \mathbf{A}'\mathbf{x}_t + \mathbf{H}'\hat{\boldsymbol{\xi}}_{t|t-1}$$

$$\mathbf{C}_{t|t-1} = \mathbf{H}'\mathbf{P}_{t|t-1}\mathbf{H} + \mathbf{R}$$

Classical econometrician: choose $\hat{\theta}$ so as to maximize log likelihood:

$$-\frac{Tn}{2}\log 2\pi - \frac{1}{2}\sum_{t=1}^{T}\log|\mathbf{C}_{t|t-1}|$$
$$-\frac{1}{2}\sum_{t=1}^{T}(\mathbf{y}_t - \hat{\mathbf{y}}_{t|t-1})'\mathbf{C}_{t|t-1}^{-1}(\mathbf{y}_t - \hat{\mathbf{y}}_{t|t-1})$$

III. Linear state-space models

A. State-space representation of a dynamic system
B. Kalman filter
C. Using the Kalman filter
   1. Estimating the unknown parameters
   2. Forecasting

Forecasting:

$$\mathbf{y}_t = \mathbf{A}'\mathbf{x}_t + \mathbf{H}'\xi_t + \mathbf{w}_t$$
$$E(\mathbf{y}_{t+j}|\Omega_t, \mathbf{x}_{t+j}, \mathbf{F}, \mathbf{Q}, \mathbf{A}, \mathbf{H}, \mathbf{R})$$
$$= \mathbf{A}'\mathbf{x}_{t+j} + \mathbf{H}'\mathbf{F}^j\hat{\xi}_{t|t}$$

MSE for $j = 1$:

$$E(\mathbf{y}_{t+1} - \hat{\mathbf{y}}_{t+1|t})(\mathbf{y}_{t+1} - \hat{\mathbf{y}}_{t+1|t})'$$
$$= \mathbf{H}'\mathbf{P}_{t+1|t}\mathbf{H} + \mathbf{R}$$

III. Linear state-space models

A. State-space representation of a dynamic system
B. Kalman filter
C. Using the Kalman filter
   1. Estimating the unknown parameters
   2. Forecasting
   3. Smoothed inference

Smoothed inference: might also want to form inference about ξ_t using all the data Ω_T:

$$\xi_t | \Omega_T \sim N(\hat{\xi}_{t|T}, \mathbf{P}_{t|T})$$

To derive formula, consider instead

$$\xi_t | \xi_{t+1}, \Omega_t \sim N(\xi^*_{t|t}, \mathbf{P}^*_{t|t})$$

Same kind of derivation as for Kalman filter establishes that

$$\xi^*_{t|t} = \hat{\xi}_{t|t} + \mathbf{J}_t(\xi_{t+1} - \hat{\xi}_{t+1|t})$$

$$\mathbf{J}_t = \mathbf{P}_{t|t}\mathbf{F}'\mathbf{P}^{-1}_{t+1|t}$$

$$\mathbf{P}^*_{t|t} = \mathbf{P}_{t|t} - \mathbf{J}_t\mathbf{F}\mathbf{P}_{t|t}$$

Generalization: what if $\mathbf{P}_{t+1|t}$ is singular?

If $\mathbf{P}_{t+1|t}$ is singular, then some linear combinations of ξ_{t+1} can be forecast perfectly from Ω_t, implying inference about ξ_t given Ω_t and these linear combinations of ξ_{t+1} is identical to inference about ξ_t given Ω_t alone.

Let ξ_t be $(r \times 1)$ and let the rank of $\mathbf{P}_{t+1|t}$ be $s \leq r$. Define the $(s \times 1)$ vector $\xi_t^{**} = \mathbf{H}^{**}\xi_t$ for an arbitrary $(s \times r)$ matrix $\mathbf{H}^{**}$ such that $\mathbf{P}_{t+1|t}^{**} \equiv \mathbf{H}^{**}\mathbf{P}_{t+1|t}\mathbf{H}^{**\prime}$ has rank s. For example, ξ_t^{**} might be the first s elements of ξ_t in which case $\mathbf{P}_{t+1|t}^{**}$ would be first s rows and columns of $\mathbf{P}_{t+1|t}$. Then $\xi_t|\xi_{t+1},\Omega_t$ has same distribution as $\xi_t|\xi_{t+1}^{**},\Omega_t$.

Generalization of previous results for singular $\mathbf{P}_{t+1|t}$:

$$\xi_{t|t}^{*} = \hat{\xi}_{t|t} + \mathbf{J}_t^{**}(\xi_{t+1}^{**} - \hat{\xi}_{t+1|t}^{**})$$
$$\mathbf{J}_t^{**} = \mathbf{P}_{t|t}(\mathbf{H}^{**}\mathbf{F})'\mathbf{P}_{t+1|t}^{**-1}$$
$$\mathbf{P}_{t|t}^{*} = \mathbf{P}_{t|t} - \mathbf{J}_t^{**}\mathbf{H}^{**}\mathbf{F}\mathbf{P}_{t|t}$$

Next suppose that, in addition to ξ_{t+1}, we had also observed $\mathbf{y}_{t+1}, \mathbf{y}_{t+2}, \ldots, \mathbf{y}_T$. This would contain no more information about ξ_t than was provided by ξ_{t+1} and Ω_t alone:

$$\xi_t | \xi_{t+1}, \Omega_T \sim N(\xi^*_{t|t}, \mathbf{P}^*_{t|t})$$

for the same $\xi^*_{t|t}, \mathbf{P}^*_{t|t}$.

And since

$$E(\xi_t | \xi_{t+1}, \Omega_T) = \hat{\xi}_{t|t} + \mathbf{J}^{**}_t (\xi^{**}_{t+1} - \hat{\xi}^{**}_{t+1|t}),$$

it follows from law of iterated expectations that

$$E(\xi_t | \Omega_T) = \hat{\xi}_{t|t} + \mathbf{J}^{**}_t (\hat{\xi}^{**}_{t+1|T} - \hat{\xi}^{**}_{t+1|t})$$

which we can calculate by iterating backwards for $t = T-1, T-2, \ldots$

Procedure to calculate smoothed inferences $\{\hat{\xi}_{t|T}\}_{t=1}^{T}$.

(1) Perform Kalman filter recursion and save the values of $\{\hat{\xi}_{t|t}, \hat{\xi}_{t+1|t}, \mathbf{P}_{t|t}, \mathbf{P}_{t+1|t}\}_{t=1}^{T}$.

(2) Calculate

$\mathbf{J}_t^{**} = \mathbf{P}_{t|t}(\mathbf{H}^{**}\mathbf{F})'(\mathbf{H}^{**}\mathbf{P}_{t+1|t}\mathbf{H}^{**\prime})^{-1}$

for $t = 1, 2, \ldots, T-1$, where $\mathbf{H}^{**}$ is an $(s \times r)$ matrix selecting the nonredundant elements of ξ_t.

(3) Calculate

$\hat{\xi}_{t|T} = \hat{\xi}_{t|t} + \mathbf{J}_t^{**}\mathbf{H}^{**}(\hat{\xi}_{t+1|T} - \hat{\xi}_{t+1|t})$

for $t = T-1$ where $\hat{\xi}_{T-1|T-1}, \hat{\xi}_{T|T}$, and $\hat{\xi}_{T|T-1}$ are all known from step (1).

(4) Evaluate

$\hat{\xi}_{t|T} = \hat{\xi}_{t|t} + \mathbf{J}_t^{**}\mathbf{H}^{**}(\hat{\xi}_{t+1|T} - \hat{\xi}_{t+1|t})$

for $t = T-2$ where right-hand variables are all known from step (3). Iterate for $t = T-3, T-4, \ldots$

The MSE's of these smoothed inferences are given by

$$E(\xi_t - \hat{\xi}_{t|T})(\xi_t - \hat{\xi}_{t|T})' = \mathbf{P}_{t|T}$$

where $\mathbf{P}_{t|T}$ can be found by iterating on

$$\mathbf{P}_{t|T} = \mathbf{P}_{t|t} + \mathbf{J}_t^{**}\mathbf{H}^{**}(\mathbf{P}_{t+1|T} - \mathbf{P}_{t+1|t})\mathbf{H}^{**\prime}\mathbf{J}_t^{**\prime}$$

backward starting from $t = T - 1$.

III. Linear state-space models

C. Using the Kalman filter

1. Estimating the unknown parameters
2. Forecasting
3. Smoothed inference
4. Time-varying parameters and missing observations

Suppose that $\mathbf{F}, \mathbf{Q}, \mathbf{A}, \mathbf{H}, \mathbf{R}$ are known functions of t (or more generally, known functions of $\mathbf{x}_t$):

$$\xi_{t+1} = \mathbf{F}_t\xi_t + \mathbf{v}_{t+1} \qquad E(\mathbf{v}_{t+1}\mathbf{v}_{t+1}') = \mathbf{Q}_t$$

$$\mathbf{y}_t = \mathbf{A}_t'\mathbf{x}_t + \mathbf{H}_t'\xi_t + \mathbf{w}_t \qquad E(\mathbf{w}_t\mathbf{w}_t') = \mathbf{R}_t$$

Then Kalman filter recursion immediately generalizes to:

$$\mathbf{P}_{t+1|t} = \mathbf{F}_t \mathbf{P}_{t|t} \mathbf{F}_t' + \mathbf{Q}_t$$

$$\mathbf{P}_{t+1|t+1} = \mathbf{P}_{t+1|t} -$$
$$\mathbf{P}_{t+1|t} \mathbf{H}_{t+1} (\mathbf{H}_{t+1}' \mathbf{P}_{t+1|t} \mathbf{H}_{t+1} + \mathbf{R}_{t+1})^{-1} \mathbf{H}_{t+1}' \mathbf{P}_{t+1|t}$$

$$\hat{\boldsymbol{\xi}}_{t+1|t} = \mathbf{F}_t \hat{\boldsymbol{\xi}}_{t|t}$$

$$\hat{\boldsymbol{\varepsilon}}_{t+1|t} = \mathbf{y}_{t+1} - \mathbf{A}_{t+1}' \mathbf{x}_{t+1} - \mathbf{H}_{t+1}' \hat{\boldsymbol{\xi}}_{t+1|t}$$

$$\hat{\boldsymbol{\xi}}_{t+1|t+1} = \hat{\boldsymbol{\xi}}_{t+1|t} +$$
$$\mathbf{P}_{t+1|t} \mathbf{H}_{t+1} (\mathbf{H}_{t+1}' \mathbf{P}_{t+1|t} \mathbf{H}_{t+1} + \mathbf{R}_{t+1})^{-1} \hat{\boldsymbol{\varepsilon}}_{t+1|t}$$

One simple trick for handling missing observations: if observation y_{it} is missing for date t, set ith rows of $\mathbf{A}_t'$ and $\mathbf{H}_t'$ to zero, take $y_{it} = 0$, set row i, col i of $\mathbf{R}_t$ to 1 and all other elements of row i or col i of $\mathbf{R}_t$ to zero.

Why it works: suppose for illustration the first r elements of $\mathbf{y}_{t+1}$ are missing.

$$\mathbf{A}_{t+1}' = \begin{bmatrix} \mathbf{0} \\ \tilde{\mathbf{A}}' \end{bmatrix} \qquad \mathbf{H}_{t+1}' = \begin{bmatrix} \mathbf{0} \\ \tilde{\mathbf{H}}' \end{bmatrix}$$

$$\mathbf{R}_{t+1} = \begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{R}} \end{bmatrix}$$

Then

$$\mathbf{H}_{t+1} = \begin{bmatrix} \mathbf{0} & \tilde{\mathbf{H}} \end{bmatrix}$$

$$\mathbf{P}_{t+1|t}\mathbf{H}_{t+1} = \begin{bmatrix} \mathbf{0} & \mathbf{P}_{t+1|t}\tilde{\mathbf{H}} \end{bmatrix}$$

$$\mathbf{H}'_{t+1}\mathbf{P}_{t+1|t}\mathbf{H}_{t+1} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{H}}'\mathbf{P}_{t+1|t}\tilde{\mathbf{H}} \end{bmatrix}$$

$$\mathbf{P}_{t+1|t}\mathbf{H}_{t+1}(\mathbf{H}'_{t+1}\mathbf{P}_{t+1|t}\mathbf{H}_{t+1} + \tilde{\mathbf{R}})^{-1}$$
$$= \begin{bmatrix} \mathbf{0} & \mathbf{P}_{t+1|t}\tilde{\mathbf{H}} \end{bmatrix} \times$$
$$\begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \left(\tilde{\mathbf{H}}'\mathbf{P}_{t+1|t}\tilde{\mathbf{H}} + \tilde{\mathbf{R}}\right)^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{0} & \mathbf{P}_{t+1|t}\tilde{\mathbf{H}}(\tilde{\mathbf{H}}'\mathbf{P}_{t+1|t}\tilde{\mathbf{H}} + \tilde{\mathbf{R}})^{-1} \end{bmatrix}$$

$$\mathbf{P}_{t+1|t}\mathbf{H}_{t+1}(\mathbf{H}'_{t+1}\mathbf{P}_{t+1|t}\mathbf{H}_{t+1} + \tilde{\mathbf{R}})^{-1}$$
$$= \begin{bmatrix} \mathbf{0} & \mathbf{P}_{t+1|t}\tilde{\mathbf{H}}(\tilde{\mathbf{H}}'\mathbf{P}_{t+1|t}\tilde{\mathbf{H}} + \tilde{\mathbf{R}})^{-1} \end{bmatrix}$$

$$\hat{\boldsymbol{\xi}}_{t+1|t+1} = \hat{\boldsymbol{\xi}}_{t+1|t} +$$
$$\mathbf{P}_{t+1|t}\mathbf{H}_{t+1}(\mathbf{H}'_{t+1}\mathbf{P}_{t+1|t}\mathbf{H}_{t+1} + \mathbf{R}_{t+1})^{-1}\hat{\boldsymbol{\varepsilon}}_{t+1|t}$$

acts as if first r elements of $\mathbf{y}_t$ weren't there

III. Linear state-space models

C. Using the Kalman filter
   1. Estimating the unknown parameters
   2. Forecasting
   3. Smoothed inference
   4. Time-varying parameters and missing observations
   5. Using mixed-frequency data as they arrive in real time

Practical problem for economic forecasters:

Different data are of different, asynchronous frequencies and are subsequently revised

Example: “Introducing the Euro-Sting: Short Term Indicator of Euro Area Growth”, Maximo Camacho and Gabriel Perez-Quiros

Assumption: there is an unobserved scalar f_t representing the monthly growth rate of real economic activity.

$\mathbf{z}_t^h = (4 \times 1)$ vector of “hard” indicators of f_t

z_{1t}^h = industrial production growth
z_{2t}^h = retail sales growth
z_{3t}^h = new industrial orders growth
z_{4t}^h = Euro area export growth

$$z_{it}^h = k_i^h + \beta_i^h f_t + u_{it}^h$$

$$z_{it}^h = k_i^h + \beta_i^h f_t + u_{it}^h$$
$$f_t = a_1 f_{t-1} + a_2 f_{t-2} + \cdots + a_6 f_{t-6} + \varepsilon_t^f$$
$$\varepsilon_t^f \sim N(0,1)$$
$$u_{it}^h = c_{i1}^h u_{i,t-1}^h + c_{i2}^h u_{i,t-2}^h + \cdots + c_{i,6}^h u_{i,t-6}^h + \varepsilon_{it}^h$$
$$\varepsilon_{it}^h \sim N(0, \sigma_{hi}^2)$$
$$\mathbf{z}_t^h = \mathbf{k}^h + \boldsymbol{\beta}^h f_t + \mathbf{u}_t^h$$
$$\mathbf{u}_t^h = \mathbf{C}_1^h \mathbf{u}_{t-1}^h + \mathbf{C}_2^h \mathbf{u}_{t-2}^h + \cdots + \mathbf{C}_6^h \mathbf{u}_{t-6}^h + \boldsymbol{\varepsilon}_t^h$$
$$\boldsymbol{\xi}_t = (f_t, \mathbf{u}_t^h, \mathbf{u}_{t-1}^h, \ldots, \mathbf{u}_{t-5}^h)'$$

Also have some "soft" survey measures intended to reflect year-over-year growth
z_{1t}^s = Belgium overall business indicator
z_{2t}^s = Euro-zone economic sentiment
z_{3t}^s = German IFO business climate
z_{4t}^s = Euro manufacturing purchasing managers index
z_{5t}^s = services PMI

$$z_{it}^s = k_i^s + \beta_i^s \sum_{j=0}^{11} f_{t-j} + u_{it}^s$$

$$u_{it}^s = c_{i1}^s u_{i,t-1}^s + c_{i2}^s u_{i,t-2}^s + \cdots + c_{i,6}^s u_{i,t-6}^s + \varepsilon_{it}^s$$

q_t = true monthly growth rate of real GDP in deviation from mean (not observed)

$$q_t = \tfrac{1}{3}\beta^q f_t + u_t^q$$

$$u_t^q = c_1^q u_{t-1}^q + c_2^q u_{t-2}^q + \cdots + c_6^q u_{t-6}^q + \varepsilon_t^q$$

Every three months we do observe a second revision of quarterly GDP growth

$$y_t^2 = k^2 + \tfrac{1}{3}q_t + \tfrac{2}{3}q_{t-1} + q_{t-2} + \tfrac{2}{3}q_{t-3} + \tfrac{1}{3}q_{t-4}$$

40 days earlier a more preliminary first revision was available

$y_t^1 = y_t^2 + e_{2t}$

20 days before that the initial "flash" estimate of GDP was released

$y_t^0 = y_t^1 + e_{1t}$

Model also uses quarterly employment growth ℓ_t.

Potential observation vector:

$\mathbf{y}_t = (y_t^2, \mathbf{z}_t^{h\prime}, \mathbf{z}_t^{s\prime}, \ell_t, y_t^1, y_t^0)'.$

Potential observation vector:

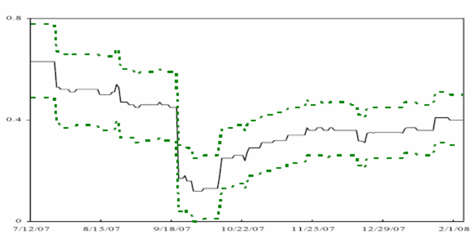
$\mathbf{y}_t = (y_t^2, \mathbf{z}_t^{h\prime}, \mathbf{z}_t^{s\prime}, \ell_t, y_t^1, y_t^0)'.$

In every month, some of these (e.g., $y_t^2, \ell_t,$ and y_t^2) are treated as missing observations

On any given day before the end of the month, a smaller subset is observed.

$$\xi_t = (f_t, f_{t-1}, \ldots, f_{t-11}, u_t^q, u_{t-1}^q, \ldots, u_{t-5}^q, \ldots$$
$$\mathbf{u}_t^{h\prime}, \ldots, \mathbf{u}_{t-5}^{h\prime}, \mathbf{u}_t^{s\prime}, \ldots, \mathbf{u}_{t-5}^{s\prime}, u_t^{\ell}, \ldots, u_{t-5}^{\ell})'$$

Model allows forecast of any variable using all information available as of any day



Real-time forecasts of 2007:Q4 real GDP growth from release of second revision on 2007/07/12 until 2008/02/13

III. Linear state-space models

A. State-space representation of a dynamic system
B. Kalman filter
C. Using the Kalman filter
D. Bayesian analysis of linear state-space models

How do we obtain θ, the vector containing unknown elements of $\mathbf{F}, \mathbf{Q}, \mathbf{A}, \mathbf{H}, \mathbf{R}$?

Classical approach: choose $\hat{\theta}$ so as to maximize log likelihood $\log p(\mathbf{Y}|\theta)$ (done by numerical search methods).

Asymptotic standard errors from

$$\hat{\theta} \approx N(\theta_0, \widehat{\mathbf{C}})$$

$$\widehat{\mathbf{C}} = \left[-\frac{\partial^2 \log p(\mathbf{Y}|\theta)}{\partial\theta\,\partial\theta'}\Big|_{\theta=\hat{\theta}} \right]^{-1}$$

Parametric bootstrap for small-sample standard errors typically infeasible (requires separate numerical optimization for each Monte Carlo draw j).

Analytical Bayesian results also unknown (problem: ξ_t is unobserved)

$$\xi_{t+1} = \mathbf{F}\xi_t + \mathbf{v}_{t+1} \qquad E(\mathbf{v}_t\mathbf{v}_t') = \mathbf{Q}$$

$$\mathbf{y}_t = \mathbf{A}'\mathbf{x}_t + \mathbf{H}'\xi_t + \mathbf{w}_t \qquad E(\mathbf{w}_t\mathbf{w}_t') = \mathbf{R}$$

Solution: Gibbs sampler

$$\xi_{t+1} = \mathbf{F}\xi_t + \mathbf{v}_{t+1} \qquad E(\mathbf{v}_t\mathbf{v}_t') = \mathbf{Q}$$
$$\mathbf{y}_t = \mathbf{A}'\mathbf{x}_t + \mathbf{H}'\xi_t + \mathbf{w}_t \qquad E(\mathbf{w}_t\mathbf{w}_t') = \mathbf{R}$$

Solution: Gibbs sampler

$\theta_1 =$ unknown elements of $\mathbf{Q}, \mathbf{R}$

$\theta_2 =$ unknown elements of $\mathbf{F}, \mathbf{A}, \mathbf{H}$

$\theta_3 =$ unknown elements of $\{\xi_0, \xi_1, \ldots, \xi_T\}$

$$\xi_{t+1} = \mathbf{F}(\theta_2)\xi_t + \mathbf{v}_{t+1} \qquad E(\mathbf{v}_t\mathbf{v}_t') = \mathbf{Q}(\theta_1)$$
$$\mathbf{y}_t = \mathbf{A}(\theta_2)'\mathbf{x}_t + \mathbf{H}(\theta_2)'\xi_t + \mathbf{w}_t$$
$$E(\mathbf{w}_t\mathbf{w}_t') = \mathbf{R}(\theta_1)$$

(1) $p(\theta_1|\theta_2, \theta_3, \mathbf{Y}, \mathbf{X})$

Knowledge of $\theta_2, \theta_3, \mathbf{Y}, \mathbf{X}$ is equivalent to direct observation of $\left\{(\mathbf{v}_{t+1}', \mathbf{w}_t')'\right\}_{t=1}^{T}$.

E.g., if priors for $\mathbf{Q}^{-1}$ and $\mathbf{R}^{-1}$ are independent $W(N_Q, \Lambda_Q)$ and $W(N_R, \Lambda_R)$, respectively, then

$$\mathbf{Q}^{-1}|\theta_2, \theta_3, \mathbf{Y}, \mathbf{X} \sim W(N_Q + T, \Lambda_Q + \mathbf{S}_Q)$$
$$\mathbf{S}_Q = \sum_{t=1}^{T} \mathbf{v}_{t+1}\mathbf{v}_{t+1}'$$
$$\mathbf{R}^{-1}|\theta_2, \theta_3, \mathbf{Y}, \mathbf{X} \sim W(N_R + T, \Lambda_R + \mathbf{S}_R)$$
$$\mathbf{S}_R = \sum_{t=1}^{T} \mathbf{w}_t\mathbf{w}_t'$$

If **Q** or **R** are singular, apply similar idea to the nonsingular subset. E.g., inverse of (1,1) element of **Q** has prior $\Gamma(N_Q, \lambda_Q)$ and posterior $\Gamma(N_Q + T, \lambda_Q + S_Q)$.

Given particular numerical values for $\boldsymbol{\theta}_2^{(j)}$ and $\boldsymbol{\theta}_3^{(j)} = ([\boldsymbol{\xi}_0^{(j)}]', [\boldsymbol{\xi}_1^{(j)}]', \ldots, [\boldsymbol{\xi}_T^{(j)}]')'$, can generate values for $\mathbf{Q}^{(j+1)}$ and $\mathbf{R}^{(j+1)}$ from these distributions.

$$\boldsymbol{\xi}_{t+1} = \mathbf{F}(\boldsymbol{\theta}_2)\boldsymbol{\xi}_t + \mathbf{v}_{t+1} \qquad E(\mathbf{v}_t\mathbf{v}_t') = \mathbf{Q}(\boldsymbol{\theta}_1)$$

$$\mathbf{y}_t = \mathbf{A}(\boldsymbol{\theta}_2)'\mathbf{x}_t + \mathbf{H}(\boldsymbol{\theta}_2)'\boldsymbol{\xi}_t + \mathbf{w}_t$$

$$E(\mathbf{w}_t\mathbf{w}_t') = \mathbf{R}(\boldsymbol{\theta}_1)$$

(2) $p(\boldsymbol{\theta}_2 | \boldsymbol{\theta}_1, \boldsymbol{\theta}_3, \mathbf{Y}, \mathbf{X})$

With knowledge of $\boldsymbol{\theta}_1, \boldsymbol{\theta}_3, \mathbf{Y}, \mathbf{X}$, this is a standard regression model.

E.g., if $\mathbf{F}$ is unrestricted,

$$\xi_{t+1} = \mathbf{F}\xi_t + \mathbf{v}_{t+1} \qquad E(\mathbf{v}_t\mathbf{v}_t') = \mathbf{Q}(\theta_1)$$

$\mathbf{f} = \mathrm{vec}(\mathbf{F}')$

prior: $\mathbf{f}|\mathbf{Q} \sim N(\mathbf{m}_F, \mathbf{Q} \otimes \mathbf{M}_F)$

posterior: $\mathbf{f}|\mathbf{Y},\mathbf{X},\theta_1,\theta_3 \sim N(\mathbf{m}_F^*, \mathbf{Q} \otimes \mathbf{M}_F^*)$

$$\mathbf{M}_F^* = \left(\mathbf{M}_F^{-1} + \textstyle\sum_{t=1}^T \xi_{t-1}\xi_{t-1}'\right)^{-1}$$

$$\mathbf{m}_F^* = (\mathbf{I}_r \otimes \mathbf{M}_F^*\mathbf{M}_F^{-1})\mathbf{m}_F + \left(\mathbf{I}_r \otimes \mathbf{M}_F^* \textstyle\sum_{t=1}^T \xi_{t-1}\xi_{t-1}'\right)\hat{\mathbf{f}}$$

$$\hat{\mathbf{f}} = \mathrm{vec}(\widehat{\mathbf{F}'})$$

$$\widehat{\mathbf{F}'} = \left(\textstyle\sum_{t=1}^T \xi_{t-1}\xi_{t-1}'\right)^{-1}\left(\textstyle\sum_{t=1}^T \xi_{t-1}\xi_t'\right)$$

So, can generate $\theta_2^{(j+1)}$ from $p(\theta_2|\theta_1^{(j+1)},\theta_3^{(j)},\mathbf{Y},\mathbf{X})$.

$$\xi_{t+1} = \mathbf{F}(\theta_2)\xi_t + \mathbf{v}_{t+1} \qquad E(\mathbf{v}_t\mathbf{v}_t') = \mathbf{Q}(\theta_1)$$
$$\mathbf{y}_t = \mathbf{A}(\theta_2)'\mathbf{x}_t + \mathbf{H}(\theta_2)'\xi_t + \mathbf{w}_t$$
$$E(\mathbf{w}_t\mathbf{w}_t') = \mathbf{R}(\theta_1)$$

(3) $p(\theta_3|\theta_1, \theta_2, \mathbf{Y}, \mathbf{X})$

With knowledge of $\theta_1, \theta_2, \mathbf{Y}, \mathbf{X}$, this is a Kalman filter problem.

$$\xi_T|\theta_1, \theta_2, \mathbf{Y}, \mathbf{X} \sim N(\hat{\xi}_{T|T}, \mathbf{P}_{T|T})$$
$$\xi_{T-1}|\xi_T, \theta_1, \theta_2, \mathbf{Y}, \mathbf{X} \sim N(\xi^*_{T-1|T-1}, \mathbf{P}^*_{T-1|T-1})$$
$$\xi^*_{T-1|T-1} = \hat{\xi}_{T-1|T-1} + \mathbf{J}_{T-1}(\xi_T - \hat{\xi}_{T|T-1})$$
$$\mathbf{J}_{T-1} = \mathbf{P}_{T-1|T-1}\mathbf{F}'\mathbf{P}^{-1}_{T|T-1}$$
$$\mathbf{P}^*_{T-1|T-1} = \mathbf{P}_{T-1|T-1} - \mathbf{J}_{T-1}\mathbf{F}\mathbf{P}_{T-1|T-1}$$

How do we generate an $(r \times 1)$ vector $\mathbf{q} \sim N(\mu, \Omega)$?

If Ω is nonsingular, find Cholesky factorization $\Omega = \Lambda\Lambda'$. Generate $\mathbf{u} \sim N(\mathbf{0}, \mathbf{I}_r)$ and $\mathbf{q} = \mu + \Lambda\mathbf{u}$.

If $\mathbf{\Omega}$ is singular, then some linear combinations of $\mathbf{q}$ are known deterministically. Set these to their known values and generate rest of $\mathbf{q}$ from non-redundant elements of distribution.

Specifically, let $\mathbf{H}$ be a known nonsingular $(r \times r)$ matrix such that

$$\mathbf{H\Omega H}' = \begin{bmatrix} \mathbf{\Omega}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

for $\mathbf{\Omega}_1$ a nonsingular $(s \times s)$ matrix $(s < r)$ with Cholesky factorization $\mathbf{\Omega}_1 = \mathbf{\Lambda}_1\mathbf{\Lambda}_1'$.

Generate

$$\mathbf{u}_1 \sim N(\mathbf{0}, \mathbf{I}_s)$$

$$\underset{r\times 1}{\mathbf{u}} = \begin{bmatrix} \mathbf{\Lambda}_1\mathbf{u}_1 \\ \mathbf{0} \end{bmatrix}$$

$$\mathbf{q} = \boldsymbol{\mu} + \mathbf{H}^{-1}\mathbf{u}$$

Generate $\xi_t^{(j+1)}$ for $t = T-1, T-2, \ldots$

$$\xi_t^{(j+1)} | \xi_{t+1}^{(j+1)}, \xi_{t+2}^{(j+1)}, \ldots, \xi_T^{(j+1)}, \boldsymbol{\theta}_1^{(j+1)}, \boldsymbol{\theta}_2^{(j+1)}, \mathbf{Y}, \mathbf{X}$$

$$\sim N(\xi_{t|t}^{*}, \mathbf{P}_{t|t}^{*})$$

$$\xi_{t|t}^{*} = \hat{\xi}_{t|t} + \mathbf{J}_t^{**} \mathbf{H}^{**} (\xi_{t+1} - \hat{\xi}_{t+1|t})$$

$$\mathbf{J}_t^{**} = \mathbf{P}_{t|t} (\mathbf{H}^{**} \mathbf{F})' (\mathbf{H}^{**} \mathbf{P}_{t+1|t} \mathbf{H}^{**})^{-1}$$

$$\mathbf{P}_{t|t}^{*} = \mathbf{P}_{t|t} - \mathbf{J}_t^{**} \mathbf{H}^{**} \mathbf{F} \mathbf{P}_{t|t}$$

So, can generate $\boldsymbol{\theta}_3^{(j+1)}$ from $p(\boldsymbol{\theta}_3 | \boldsymbol{\theta}_1^{(j+1)}, \boldsymbol{\theta}_2^{(j+1)}, \mathbf{Y}, \mathbf{X})$.

Gibbs sampler:

(1) Start with arbitrary initial guesses for $\boldsymbol{\theta}_1^{(1)}, \boldsymbol{\theta}_2^{(1)}, \boldsymbol{\theta}_3^{(1)}$ (e.g., $\boldsymbol{\theta}_3^{(1)} = \mathbf{0}$).

(2) Generate:

$\boldsymbol{\theta}_1^{(j+1)}$ from $p(\boldsymbol{\theta}_1 | \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta}_2^{(j)}, \boldsymbol{\theta}_3^{(j)})$

$\boldsymbol{\theta}_2^{(j+1)}$ from $p(\boldsymbol{\theta}_2 | \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta}_1^{(j+1)}, \boldsymbol{\theta}_3^{(j)})$

$\boldsymbol{\theta}_3^{(j+1)}$ from $p(\boldsymbol{\theta}_3 | \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta}_1^{(j+1)}, \boldsymbol{\theta}_2^{(j+1)})$

for $j = 1, 2, \ldots, D$

From these one can calculate such things as:

(1) Small-sample confidence intervals for elements of $\mathbf{F}$ (e.g., for 95% of values of j between D_0 and D the $(1,1)$ element of $\mathbf{F}(\theta_2^{(j)})$ is between a_1 and a_2).

(2) Best guess as to state of business cycle C_t at historical date t (average value of $(1,1)$ element of $\xi_t^{(j)}$ for j between D_0 and D) and uncertainty about this guess (standard deviation of $\xi_t^{(j)}$), where uncertainty incorporates both filter uncertainty,

$$\mathbf{P}_{t|T}(\theta_1,\theta_2) \neq \mathbf{0},$$

and uncertainty about parameter values:

θ_1, θ_2 unknown.

(3) Optimal forecast $\hat{\mathbf{y}}_{t+m|t}$, incorporating uncertainty about parameter values, or average value of

$$[\mathbf{A}(\theta_2^{(j)})]'\mathbf{x}_{t+m} + [\mathbf{H}(\theta_2^{(j)})]'[\mathbf{F}(\theta_2^{(j)})]^m\hat{\xi}_{t|t}^{(j)}$$