III. Linear state-space models

A. State-space representation of a dynamic system
B. Kalman filter
C. Using the Kalman filter
D. Bayesian analysis of linear state-space models
E. Solutions to linear rational expectations models
   1. Problem statement

$$\mathbf{A}E_t\mathbf{y}_{t+1} = \mathbf{B}\mathbf{y}_t + \mathbf{C}\mathbf{x}_t$$

$$\mathbf{x}_{t+1} = \mathbf{\Phi}\mathbf{x}_t + \boldsymbol{\varepsilon}_{t+1}$$

$\mathbf{x}_t$ exogenous

---

e.g., if $\mathbf{z}_t = (k \times 1)$

$$\mathbf{z}_t = \boldsymbol{\phi}_1\mathbf{z}_{t-1} + \boldsymbol{\phi}_2\mathbf{z}_{t-2} + \cdots + \boldsymbol{\phi}_p\mathbf{z}_{t-p} + \mathbf{v}_t$$

$$\underset{(kp\times 1)}{\mathbf{x}_t} = \begin{bmatrix} \mathbf{z}_t \\ \mathbf{z}_{t-1} \\ \vdots \\ \mathbf{z}_{t+p+1} \end{bmatrix}$$

$$\mathbf{\Phi} = \begin{bmatrix} \boldsymbol{\phi}_1 & \boldsymbol{\phi}_2 & \cdots & \boldsymbol{\phi}_{p-1} & \boldsymbol{\phi}_p \\ \mathbf{I}_k & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}_k & \mathbf{0} \end{bmatrix}$$

$$\mathbf{y}_t = \begin{bmatrix} \underset{(n_k\times 1)}{\mathbf{k}_t} \\ \underset{(n_d\times 1)}{\mathbf{d}_t} \end{bmatrix}$$

$\mathbf{k}_t$ = predetermined

(chosen by agents at $t-1$)

$\mathbf{k}_t = h(\mathbf{x}_{t-1}, \mathbf{x}_{t-2}, \ldots \mathbf{x}_1, \mathbf{k}_0)$

$d_t = m(\mathbf{x}_t, \mathbf{x}_{t-1}, \ldots, \mathbf{x}_1, \mathbf{k}_0)$

goal of solution method:

find $h(.)$ and $m(.)$

E. Solutions to linear rational expectations models
   1. Problem statement
   2. Blanchard-Kahn solution method

$\mathbf{A}E_t\mathbf{y}_{t+1} = \mathbf{B}\mathbf{y}_t + \mathbf{C}\mathbf{x}_t$

This method assumes $\mathbf{A}$ is nonsingular.

Drawback: if original system involves $E_t\mathbf{z}_{t+2}$ or $\mathbf{z}_{t-2}$, can be written in canonical form using companion form, but at cost of making $\mathbf{A}$ singular.

$\mathbf{A}E_t\mathbf{y}_{t+1} = \mathbf{B}\mathbf{y}_t + \mathbf{C}\mathbf{x}_t$

Find Jordan form of $\mathbf{A}^{-1}\mathbf{B}$:

$\mathbf{A}^{-1}\mathbf{B} = \mathbf{V}^{-1}\mathbf{J}\mathbf{V}$

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{J}_s \end{bmatrix}$$

$$\mathbf{J}_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}$$

Order eigenvalues λ_i such that first n_s are less than or equal to unity in modulus and next n_u are greater than unity in modulus. Assumption for unique stationary solution: $n_s = n_k$

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_u \end{bmatrix}$$

diagonal elements of $\mathbf{J}_s$ are all ≤ 1 in modulus

diagonal elements of $\mathbf{J}_u$ are all > 1 in modulus

premultiply original system

$$\mathbf{A}E_t\mathbf{y}_{t+1} = \mathbf{B}\mathbf{y}_t + \mathbf{C}\mathbf{x}_t$$

by $\mathbf{V}\mathbf{A}^{-1}$

$$\mathbf{V}E_t\mathbf{y}_{t+1} = \mathbf{V}\mathbf{A}^{-1}\mathbf{B}\mathbf{y}_t + \mathbf{V}\mathbf{A}^{-1}\mathbf{C}\mathbf{x}_t$$

$$E_t\mathbf{V}\mathbf{y}_{t+1} = \mathbf{J}\mathbf{V}\mathbf{y}_t + \mathbf{C}^*\mathbf{x}_t$$

$$E_t\mathbf{V}\mathbf{y}_{t+1} = \mathbf{J}\mathbf{V}\mathbf{y}_t + \mathbf{C}^*\mathbf{x}_t$$

define

$$\begin{bmatrix} \underset{(n_s\times 1)}{\mathbf{s}_t} \\ \underset{(n_d\times 1)}{\mathbf{u}_t} \end{bmatrix} = \mathbf{V}\mathbf{y}_t$$

$$\begin{bmatrix} E_t\mathbf{s}_{t+1} \\ E_t\mathbf{u}_{t+1} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_s\mathbf{s}_t \\ \mathbf{J}_u\mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \mathbf{C}^*_{sx}\mathbf{x}_t \\ \mathbf{C}^*_{ux}\mathbf{x}_t \end{bmatrix}$$

$$E_t\mathbf{u}_{t+1} = \mathbf{J}_u\mathbf{u}_t + \mathbf{C}^*_{ux}\mathbf{x}_t$$

$$\mathbf{u}_t = \mathbf{J}_u^{-1}E_t\mathbf{u}_{t+1} - \mathbf{J}_u^{-1}\mathbf{C}^*_{ux}\mathbf{x}_t$$

$$\mathbf{u}_{t+1} = \mathbf{J}_u^{-1}E_{t+1}\mathbf{u}_{t+2} - \mathbf{J}_u^{-1}\mathbf{C}^*_{ux}\mathbf{x}_{t+1}$$

$$E_t\mathbf{u}_{t+1} = \mathbf{J}_u^{-1}E_t\mathbf{u}_{t+2} - \mathbf{J}_u^{-1}\mathbf{C}^*_{ux}E_t\mathbf{x}_{t+1}$$

$$\mathbf{u}_t = -\mathbf{J}_u^{-1}\mathbf{C}^*_{ux}\mathbf{x}_t - \mathbf{J}_u^{-2}\mathbf{C}^*_{ux}E_t\mathbf{x}_{t+1} + \mathbf{J}_u^{-2}E_t\mathbf{u}_{t+2}$$

$$\mathbf{u}_t = -\mathbf{J}_u^{-1}\sum_{h=0}^{\infty}\mathbf{J}_u^{-h}\mathbf{C}^*_{ux}E_t\mathbf{x}_{t+h}$$

$$\begin{bmatrix} E_t\mathbf{s}_{t+1} \\ E_t\mathbf{u}_{t+1} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_s\mathbf{s}_t \\ \mathbf{J}_u\mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \mathbf{C}^*_{sx}\mathbf{x}_t \\ \mathbf{C}^*_{ux}\mathbf{x}_t \end{bmatrix}$$

$$\begin{aligned} \mathbf{u}_t &= -\mathbf{J}_u^{-1}\sum_{h=0}^{\infty}\mathbf{J}_u^{-h}\mathbf{C}^*_{ux}E_t\mathbf{x}_{t+h} \\ &= -\mathbf{J}_u^{-1}\sum_{h=0}^{\infty}\mathbf{J}_u^{-h}\mathbf{C}^*_{ux}\mathbf{\Phi}^h\mathbf{x}_t \\ &= \mathbf{H}_{ux}\mathbf{x}_t \end{aligned}$$

Note if

$$\mathbf{S} = \mathbf{J}_u^{-1}\sum_{h=0}^{\infty}\mathbf{J}_u^{-h}\mathbf{C}_{ux}^{*}\mathbf{\Phi}^{h}$$

then

$$\mathbf{S} - \mathbf{J}_u^{-1}\mathbf{S}\mathbf{\Phi} = \mathbf{J}_u^{-1}\mathbf{C}_{ux}^{*}$$

$$\text{vec}(\mathbf{S}) - (\mathbf{\Phi}' \otimes \mathbf{J}_u^{-1})\text{vec}(\mathbf{S}) = \text{vec}(\mathbf{J}_u^{-1}\mathbf{C}_{ux}^{*})$$

$$\text{vec}(\mathbf{S}) = [\mathbf{I}_{n_u^2} - \mathbf{\Phi}' \otimes \mathbf{J}_u^{-1}]^{-1}\text{vec}(\mathbf{J}_u^{-1}\mathbf{C}_{ux}^{*})$$

$$\mathbf{u}_t = -\mathbf{J}_u^{-1}\sum_{h=0}^{\infty}\mathbf{J}_u^{-h}\mathbf{C}_{ux}^{*}\mathbf{\Phi}^{h}\mathbf{x}_t = \mathbf{H}_{ux}\mathbf{x}_t$$

$$\mathbf{H}_{ux} = -[\mathbf{I}_{n_u^2} - \mathbf{\Phi}' \otimes \mathbf{J}_u^{-1}]^{-1}\text{vec}(\mathbf{J}_u^{-1}\mathbf{C}_{ux}^{*})$$

$$\begin{bmatrix} \mathbf{s}_t \\ \mathbf{u}_t \end{bmatrix} = \mathbf{V}\mathbf{y}_t = \mathbf{V}\begin{bmatrix} \mathbf{k}_t \\ \mathbf{d}_t \end{bmatrix}$$

$$\mathbf{s}_t = \mathbf{V}_{sk}\mathbf{k}_t + \mathbf{V}_{sd}\mathbf{d}_t$$

$$\mathbf{u}_t = \mathbf{V}_{uk}\mathbf{k}_t + \mathbf{V}_{ud}\mathbf{d}_t$$

$\mathbf{u}_t = \mathbf{V}_{uk}\mathbf{k}_t + \mathbf{V}_{ud}\mathbf{d}_t$

suppose that $\mathbf{V}_{ud}^{-1}$ exists (required to be able to choose $\mathbf{d}_t$ so as to eliminate unstable eigenvalues)

$\mathbf{d}_t = \mathbf{V}_{ud}^{-1}\mathbf{u}_t - \mathbf{V}_{ud}^{-1}\mathbf{V}_{uk}\mathbf{k}_t$

$$\mathbf{d}_t = \mathbf{V}_{ud}^{-1}\mathbf{u}_t - \mathbf{V}_{ud}^{-1}\mathbf{V}_{uk}\mathbf{k}_t$$

$$\mathbf{s}_t = \mathbf{V}_{sk}\mathbf{k}_t + \mathbf{V}_{sd}\mathbf{d}_t$$

$$\mathbf{s}_t = \mathbf{V}_{sk}\mathbf{k}_t + \mathbf{V}_{sd}[\mathbf{V}_{ud}^{-1}\mathbf{u}_t - \mathbf{V}_{ud}^{-1}\mathbf{V}_{uk}\mathbf{k}_t]$$

Also define $\mathbf{R} = \mathbf{V}^{-1}$

$$\begin{bmatrix} \mathbf{s}_t \\ \mathbf{u}_t \end{bmatrix} = \mathbf{V}\begin{bmatrix} \mathbf{k}_t \\ \mathbf{d}_t \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{k}_t \\ \mathbf{d}_t \end{bmatrix} = \mathbf{R}\begin{bmatrix} \mathbf{s}_t \\ \mathbf{u}_t \end{bmatrix}$$

$$\mathbf{k}_t = \mathbf{R}_{ks}\mathbf{s}_t + \mathbf{R}_{ku}\mathbf{u}_t$$

$$\begin{aligned}
\mathbf{k}_t &= \mathbf{R}_{ks}\mathbf{s}_t + \mathbf{R}_{ku}\mathbf{u}_t \\
\mathbf{k}_{t+1} &= \mathbf{R}_{ks}\mathbf{s}_{t+1} + \mathbf{R}_{ku}\mathbf{u}_{t+1} \\
E_t\mathbf{k}_{t+1} &= \mathbf{R}_{ks}E_t\mathbf{s}_{t+1} + \mathbf{R}_{ku}E_t\mathbf{u}_{t+1} \\
&= \mathbf{k}_{t+1}
\end{aligned}$$

(b/c $\mathbf{k}_{t+1}$ is determined at t)

Recall

$$\begin{bmatrix} E_t\mathbf{s}_{t+1} \\ E_t\mathbf{u}_{t+1} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_s\mathbf{s}_t \\ \mathbf{J}_u\mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \mathbf{C}^*_{sx}\mathbf{x}_t \\ \mathbf{C}^*_{ux}\mathbf{x}_t \end{bmatrix}$$

$$\begin{aligned}
\mathbf{k}_{t+1} &= \mathbf{R}_{ks}E_t\mathbf{s}_{t+1} + \mathbf{R}_{ku}E_t\mathbf{u}_{t+1} \\
\mathbf{k}_{t+1} &= \mathbf{R}_{ks}(\mathbf{J}_s\mathbf{s}_t + \mathbf{C}^*_{sx}\mathbf{x}_t) \\
&\quad + \mathbf{R}_{ku}(\mathbf{J}_u\mathbf{u}_t + \mathbf{C}^*_{ux}\mathbf{x}_t)
\end{aligned}$$

$$\begin{aligned}
\mathbf{k}_{t+1} &= \mathbf{R}_{ks}(\mathbf{J}_s\mathbf{s}_t + \mathbf{C}^*_{sx}\mathbf{x}_t) \\
&\quad + \mathbf{R}_{ku}(\mathbf{J}_u\mathbf{u}_t + \mathbf{C}^*_{ux}\mathbf{x}_t)
\end{aligned}$$

Recall also

$$\mathbf{s}_t = \mathbf{V}_{sk}\mathbf{k}_t + \mathbf{V}_{sd}[\mathbf{V}^{-1}_{ud}\mathbf{u}_t - \mathbf{V}^{-1}_{ud}\mathbf{V}_{uk}\mathbf{k}_t]$$

so

$$\begin{aligned}
\mathbf{k}_{t+1} &= \mathbf{R}_{ks}\mathbf{J}_s\{\mathbf{V}_{sk}\mathbf{k}_t + \mathbf{V}_{sd}[\mathbf{V}^{-1}_{ud}\mathbf{u}_t - \mathbf{V}^{-1}_{ud}\mathbf{V}_{uk}\mathbf{k}_t]\} \\
&\quad + \mathbf{R}_{ks}\mathbf{C}^*_{sx}\mathbf{x}_t + \mathbf{R}_{ku}(\mathbf{J}_u\mathbf{u}_t + \mathbf{C}^*_{ux}\mathbf{x}_t)
\end{aligned}$$

Finally, since

$$
\begin{aligned}
\mathbf{u}_t &= \mathbf{H}_{ux}\mathbf{x}_t \\
\mathbf{H}_{ux} &= -[\mathbf{I}_{n_u^2} - \mathbf{\Phi}' \otimes \mathbf{J}_u^{-1}]^{-1}\text{vec}(\mathbf{J}_u^{-1}\mathbf{C}_{ux}^*) \\
\mathbf{k}_{t+1} &= \mathbf{H}_{kk}\mathbf{k}_t + \mathbf{H}_{kx}\mathbf{x}_t \\
\mathbf{H}_{kk} &= \mathbf{R}_{ks}\mathbf{J}_s\mathbf{V}_{sk} - \mathbf{R}_{ks}\mathbf{J}_s\mathbf{V}_{sd}\mathbf{V}_{ud}^{-1}\mathbf{V}_{uk} \\
\mathbf{H}_{kx} &= \mathbf{R}_{ks}\mathbf{J}_s\mathbf{V}_{sd}\mathbf{V}_{ud}^{-1}\mathbf{H}_{ux} + \mathbf{R}_{ks}\mathbf{C}_{sx}^* \\
&\quad + \mathbf{R}_{ku}\mathbf{J}_u\mathbf{H}_{ux} + \mathbf{R}_{ku}\mathbf{C}_{ux}^*
\end{aligned}
$$

From our earlier expression

$$\mathbf{d}_t = \mathbf{V}_{ud}^{-1}\mathbf{u}_t - \mathbf{V}_{ud}^{-1}\mathbf{V}_{uk}\mathbf{k}_t$$

we likewise have

$$\mathbf{d}_t = \mathbf{V}_{ud}^{-1}\mathbf{H}_{ux}\mathbf{x}_t - \mathbf{V}_{ud}^{-1}\mathbf{V}_{uk}\mathbf{k}_t$$

Conclusion: the system

$$
\begin{aligned}
\mathbf{A}E_t\mathbf{y}_{t+1} &= \mathbf{B}\mathbf{y}_t + \mathbf{C}\mathbf{x}_t \\
\mathbf{x}_{t+1} &= \mathbf{\Phi}\mathbf{x}_t + \boldsymbol{\varepsilon}_{t+1} \\
\mathbf{y}_t &= \begin{bmatrix} \mathbf{k}_t \\ \mathbf{d}_t \end{bmatrix}
\end{aligned}
$$

has the solution

$$\mathbf{k}_{t+1} = \mathbf{H}_{kk}\mathbf{k}_t + \mathbf{H}_{kx}\mathbf{x}_t$$

$$\mathbf{d}_t = \mathbf{V}_{ud}^{-1}\mathbf{H}_{ux}\mathbf{x}_t - \mathbf{V}_{ud}^{-1}\mathbf{V}_{uk}\mathbf{k}_t$$

$$\mathbf{x}_{t+1} = \mathbf{\Phi}\mathbf{x}_t + \boldsymbol{\varepsilon}_{t+1}$$

which is a state-space system with $\xi_{t+1} = (\mathbf{x}_{t+1}', \mathbf{k}_{t+1}')'$

E. Solutions to linear rational expectations models
   1. Problem statement
   2. Blanchard-Kahn solution method
   3. Klein solution method

$$\mathbf{A}E_t\mathbf{y}_{t+1} = \mathbf{B}\mathbf{y}_t + \mathbf{C}\mathbf{x}_t$$

$$\mathbf{x}_{t+1} = \mathbf{\Phi}\mathbf{x}_t + \boldsymbol{\varepsilon}_{t+1}$$

More general case:

$\mathbf{A}$ is singular

As long as there exists a
(possibly complex) scalar z
such that $|\mathbf{A}z - \mathbf{B}| \neq 0$
then there exist (possibly complex)
matrices $\mathbf{Q}$ and $\mathbf{Z}$ such that
(i) $\mathbf{Q}^H\mathbf{Q} = \mathbf{I}_n$
where $\mathbf{Q}^H$ means transpose
and take complex conjugates

(ii) $\mathbf{Z}^H\mathbf{Z} = \mathbf{I}_n$
(iii) $\mathbf{QAZ} = \mathbf{S}$ is upper triangular
(iv) $\mathbf{QBZ} = \mathbf{T}$ is upper triangular
(v) variables can be ordered
so that t_{ii}/s_{ii} are increasing
in modulus (with any zero s_{ii}
appearing last)

Called complex generalized
Schur form
Assume the first n_s values of
t_{ii}/s_{ii} are < 1 and last n_u are > 1

premultiply original system

$$\mathbf{A}E_t\mathbf{y}_{t+1} = \mathbf{B}\mathbf{y}_t + \mathbf{C}\mathbf{x}_t$$

by $\mathbf{Q}$

$$E_t\mathbf{QAZZ}^H\mathbf{y}_{t+1} = \mathbf{QBZZ}^H\mathbf{y}_t + \mathbf{QC}\mathbf{x}_t$$

$$E_t\mathbf{QAZZ}^H\mathbf{y}_{t+1} = \mathbf{QBZZ}^H\mathbf{y}_t + \mathbf{QC}\mathbf{x}_t$$

$$\mathbf{QAZ} = \mathbf{S} \qquad \mathbf{QBZ} = \mathbf{T}$$

$$E_t\mathbf{SZ}^H\mathbf{y}_{t+1} = \mathbf{TZ}^H\mathbf{y}_t + \mathbf{QC}\mathbf{x}_t$$

$$E_t\mathbf{SZ}^H\mathbf{y}_{t+1} = \mathbf{TZ}^H\mathbf{y}_t + \mathbf{QC}\mathbf{x}_t$$

Define

$$\begin{bmatrix} \mathbf{s}_t \\ \mathbf{u}_t \end{bmatrix} = \mathbf{Z}^H\mathbf{y}_t$$

$$\mathbf{S}\begin{bmatrix} E_t\mathbf{s}_{t+1} \\ E_t\mathbf{u}_{t+1} \end{bmatrix} = \mathbf{T}\begin{bmatrix} \mathbf{s}_t \\ \mathbf{u}_t \end{bmatrix} + \mathbf{C}^*\mathbf{x}_t$$

$$\begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{0} & \mathbf{S}_{22} \end{bmatrix} \begin{bmatrix} E_t \mathbf{s}_{t+1} \\ E_t \mathbf{u}_{t+1} \end{bmatrix} = \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{0} & \mathbf{T}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{s}_t \\ \mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \mathbf{C}^*_{sx} \mathbf{x}_t \\ \mathbf{C}^*_{ux} \mathbf{x}_t \end{bmatrix}$$

where $\mathbf{T}_{22}$ is invertible by construction

$$\begin{aligned} \mathbf{S}_{22} E_t \mathbf{u}_{t+1} &= \mathbf{T}_{22} \mathbf{u}_t + \mathbf{C}^*_{ux} \mathbf{x}_t \\ \mathbf{u}_t &= -\mathbf{T}_{22}^{-1} \sum_{h=0}^{\infty} [\mathbf{T}_{22}^{-1} \mathbf{S}_{22}]^h \mathbf{C}^*_{ux} E_t \mathbf{x}_{t+h} \\ &= -\mathbf{T}_{22}^{-1} \sum_{h=0}^{\infty} [\mathbf{T}_{22}^{-1} \mathbf{S}_{22}]^h \mathbf{C}^*_{ux} \mathbf{\Phi}^h \mathbf{x}_t \\ &= \mathbf{H}_{ux} \mathbf{x}_t \\ \mathbf{H}_{ux} &= [(\mathbf{\Phi}' \otimes \mathbf{S}_{22}) - (\mathbf{I}_{n_x} \otimes \mathbf{T}_{22})]^{-1} \text{vec}(\mathbf{C}^*_{ux}) \end{aligned}$$

Can then use parallel calculations to those for Blanchard-Kahn to arrive at

$$\begin{aligned} \mathbf{k}_{t+1} &= \mathbf{H}_{kk} \mathbf{k}_t + \mathbf{H}_{kx} \mathbf{x}_t \\ \mathbf{d}_t &= \mathbf{V}_{ud}^{-1} \mathbf{H}_{ux} \mathbf{x}_t - \mathbf{V}_{ud}^{-1} \mathbf{V}_{uk} \mathbf{k}_t \\ \mathbf{x}_{t+1} &= \mathbf{\Phi} \mathbf{x}_t + \boldsymbol{\varepsilon}_{t+1} \end{aligned}$$