

### III. Linear state-space models

- A. State-space representation of a dynamic system
- B. Kalman filter
- C. Using the Kalman filter
- D. Bayesian analysis of linear state-space models
- E. Solutions to linear rational expectations models
- F. Estimating DSGE models

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instantaneous utility function:

$$U_t = a_t^b \left[ \frac{1}{1-\lambda_c} (C_t - hC_{t-1})^{1-\lambda_c} - \frac{a_t^l}{1+\lambda_l} (\ell_t)^{1+\lambda_l} \right]$$

$h > 0 \Rightarrow$  habit persistence

$$a_t^b = \rho_b a_{t-1}^b + \eta_t^b \quad \eta_t^b \sim \text{i.i.d. } N(0, \sigma_b^2)$$

$\Rightarrow$  shock to intertemporal subs

$$a_t^l = \rho_L a_{t-1}^l + \eta_t^l$$

$\Rightarrow$  shock to intratemporal subs

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Let  $\hat{C}_t$  denote deviation of  $\log(C_t)$  from its steady-state value

$$(1) \hat{C}_t = \left( \frac{h}{1+h} \right) \hat{C}_{t-1} + \left( \frac{1}{1+h} \right) E_t \hat{C}_{t+1}$$

$$- \frac{(1-h)}{(1+h)\lambda_c} (\hat{R}_t - E_t \hat{\pi}_{t+1})$$

$$+ \frac{(1-h)}{(1+h)\lambda_c} (\hat{a}_t^b - E_t \hat{a}_{t+1}^b)$$

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capital evolution:

$$K_t = K_{t-1}(1 - \delta) + [1 - S(a_t^I I_t / I_{t-1})] I_t$$

$S(\cdot)$  = adjustment costs

$$a_t^I = \rho_I a_{t-1}^I + \eta_t^I$$

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$$(2) \hat{K}_t = (1 - \delta)\hat{K}_{t-1} + \delta\hat{I}_{t-1}$$

$$(3) \hat{I}_t = \left(\frac{1}{1+\beta}\right)\hat{I}_{t-1} + \left(\frac{\beta}{1+\beta}\right)E_t\hat{I}_{t+1} \\ + \frac{\varphi}{1+\beta}\hat{Q}_t - \frac{\beta E_t a_{t+1}^I - a_t^I}{1+\beta}$$

$$\varphi = 1/S''$$

$Q_t$  = value of capital stock

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$$(4) \hat{Q}_t = -(\hat{R}_t - \hat{\pi}_{t+1}) + \frac{1-\delta}{1-\delta+\bar{r}_k} E_t \hat{Q}_{t+1} \\ + \frac{\bar{r}_k}{1-\delta+\bar{r}_k} E_t \hat{r}_{K,t+1} + \eta_t^Q$$

$r_{Kt}$  = rate of return to capital

$\eta_t^Q$  = tacked on

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output from producer of  
intermediate good of type  $j$

$$y_t^j = a_t^a K_{t-1}(j)^\alpha L_t(j)^{1-\alpha} - \Phi$$

$\Phi$  = fixed cost

$a_t^a$  = productivity shock

$$a_t^a = \rho_a a_{t-1}^a + \eta_t^a$$

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$L_t(j)$  = aggregate of labor  
hired from each household  $\tau$

$$L_t(j) = \left\{ \int_0^1 [\ell_t(\tau)]^{1/(1+\lambda_{w,t})} d\tau \right\}^{1+\lambda_{w,t}}$$

$$\lambda_{w,t} = \lambda_w + \eta_t^w$$

$\eta_t^w$  = shock to workers' market  
power

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wage stickiness:

a fraction  $\xi_w$  of workers are  
not allowed to change their wage  
but instead have their wage increase  
from the previous value by

$$(P_{t-1}/P_{t-2})^{\gamma_w}$$

$\gamma_w$  = degree of indexing

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$$\begin{aligned}
 (5) \quad \hat{w}_t &= \frac{\beta}{1+\beta} E\hat{w}_{t+1} + \frac{1}{1+\beta} \hat{w}_{t-1} \\
 &+ \frac{\beta}{1+\beta} E_t \hat{\pi}_{t+1} - \frac{1+\beta\gamma_w}{1+\beta} \hat{\pi}_t + \frac{\gamma_w}{1+\beta} \hat{\pi}_{t-1} \\
 &- h_w \left[ \hat{w}_t - \lambda_L \hat{L}_t - \frac{\lambda_c}{1-h} (\hat{C}_t - h\hat{C}_{t-1}) \right. \\
 &\quad \left. - \hat{a}_t^L - \hat{\eta}_t^w \right] \\
 h_w &= \frac{1}{1+\beta} \frac{(1-\beta\xi_w)(1-\xi_w)}{[1+(1+\lambda_w)\lambda_L/\lambda_w]\xi_w}
 \end{aligned}$$

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labor demand

$$\frac{W_t L_t(j)}{r_{K,t} z_t K_{t-1}(j)} = \frac{1-\alpha}{\alpha}$$

$z_t$  = capital utilization

$$(6) \quad \hat{L}_t = -\hat{w}_t + (1 + \psi) \hat{r}_{K,t} + \hat{K}_{t-1}$$

$\psi$  = parameter based on cost of utilizing capital

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intermediate goods sold to final goods producer with market power of firm  $j$  governed by

$$\lambda_{p,t} = \lambda_p + \eta_t^p$$

$\xi_p$  = fraction allowed to adjust prices

$$\gamma_p = \text{indexing parameter}$$

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$$(7) \hat{\pi}_t = \frac{\beta}{1+\beta\gamma_p} E_t \hat{\pi}_{t+1} + \frac{\gamma_p}{1+\beta\gamma_p} \hat{\pi}_{t-1} + h_p \left[ \alpha \hat{r}_{K,t} + (1-\alpha) \hat{w}_t - \hat{a}_t^a + \eta_t^p \right]$$

$$h_p = \frac{1}{1+\beta\gamma_p} \frac{(1-\beta\xi_p)(1-\xi_p)}{\xi_p}$$

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goods market equilibrium

$$(8) \hat{Y}_t = [1 - \delta(\bar{K}/\bar{Y}) - (\bar{G}/\bar{Y})] \hat{C}_t + \delta(\bar{K}/\bar{Y}) \hat{I}_t + (\bar{G}/\bar{Y}) \hat{a}_t^G$$

production function then determines  $r_{K,t}$

$$(9) \hat{Y}_t = \phi \hat{a}_t^a + \phi \alpha \hat{K}_{t-1} + \phi \alpha \psi \hat{r}_{K,t} + \phi(1-\alpha) \hat{L}_t$$

$$\phi = 1 + \frac{\Phi}{\text{s.s. costs}}$$

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monetary policy (Taylor Rule)

$$(10) \hat{R}_t = \rho \hat{R}_{t-1} + (1-\rho) \{ \bar{\pi}_t + r_\pi (\hat{\pi}_{t-1} - \bar{\pi}_{t-1}) + r_Y (\hat{Y}_t - \hat{Y}_t^p) \} + r_{\Delta\pi} (\hat{\pi}_t - \hat{\pi}_{t-1}) + r_{\Delta Y} [\hat{Y}_t - \hat{Y}_t^p - (\hat{Y}_{t-1} - \hat{Y}_{t-1}^p)] + \eta_t^R$$

$\bar{\pi}_t =$  inflation target

$$\bar{\pi}_t = \rho_\pi \bar{\pi}_{t-1} + \eta_t^\pi$$

$\hat{Y}_t^p =$  output level if prices perfectly flexible

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$$\mathbf{y}_t = (\hat{C}_t, \hat{C}_{t-1}, \hat{R}_t, \hat{R}_{t-1}, \hat{K}_t, \hat{K}_{t-1}, \hat{I}_t, \hat{I}_{t-1}, \hat{Q}_t, \hat{w}_t, \hat{w}_{t-1}, \hat{L}_t, \hat{\pi}_t, \hat{\pi}_{t-1}, \hat{Y}_t, \hat{r}_{K,t})'$$

$$\mathbf{x}_t = (\hat{a}_t^b, \hat{a}_t^I, \eta_t^O, \hat{a}_t^L, \eta_t^w, \hat{a}_t^a, \eta_t^p, \hat{a}_t^G, \bar{\pi}_t, \eta_t^R)'$$

equations (1)-(10) (along with lag definitions) can be written as

$$\mathbf{A}E_t\mathbf{y}_{t+1} = \mathbf{B}\mathbf{y}_t + \mathbf{C}\mathbf{x}_t$$

while shocks satisfy

$$\mathbf{x}_{t+1} = \mathbf{\Phi}\mathbf{x}_t + \boldsymbol{\varepsilon}_{t+1}$$

(note also  $E_t\mathbf{x}_{t+1} = \mathbf{\Phi}\mathbf{x}_t$ )

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Observed data:

OLS regression of log real consumption on constant and time trend

$$\text{residual} = z_{1t} = \hat{C}_t$$

Same for log of investment yields  $\hat{I}_t$

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Other data: GDP, real wages, GDP deflator, nominal interest rate  
Treat  $\hat{Q}_t, \hat{r}_{K,t}, \hat{a}_t^a$  as unobservable

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state equation:

$$\xi_{t+1} = \mathbf{F}\xi_t + \mathbf{v}_{t+1}$$

(came from solution to DSGE model)

observation equation:

$$\mathbf{z}_t = \mathbf{H}'\xi_t$$

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Let  $\theta$  = parameters of structural model

Use Kalman filter to evaluate

likelihood function

$$p(\mathbf{z}_1, \dots, \mathbf{z}_T | \theta)$$

prior  $p(\theta)$

$$\text{posterior } p(\theta | \mathbf{Z}) \propto p(\theta)p(\mathbf{Z} | \theta)$$

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prior  $p(\theta)$

$$\text{posterior } p(\theta | \mathbf{Z}) \propto p(\theta)p(\mathbf{Z} | \theta)$$

we can sample from posterior

using method such as Metropolis-

Hastings

$$\lambda(\theta) = \log p(\theta) + \log p(\mathbf{Z} | \theta)$$

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$$\lambda(\boldsymbol{\theta}) = \log p(\boldsymbol{\theta}) + \log p(\mathbf{Z}|\boldsymbol{\theta})$$

(1) Find mode of posterior distribution using numerical optimization

$$\boldsymbol{\theta}^* = \arg \max \lambda(\boldsymbol{\theta})$$

(2) Find Hessian of posterior distribution

$$\mathbf{H}(\boldsymbol{\theta}^*) = -\frac{\partial^2 \lambda(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}^*}$$

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(3) Set  $\boldsymbol{\theta}^{(j)}$  = arbitrary starting value for  $j = 1$

(4) Generate  $\tilde{\boldsymbol{\theta}}^{(j+1)} \sim N(\boldsymbol{\theta}^{(j)}, c\mathbf{H}(\boldsymbol{\theta}^*))$   
for choice of  $c$  described below  
(could experiment to see if Student  $t$  works better)

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(5) Generate  $u^{(j+1)} \sim U[0, 1]$

(6) Set  $\boldsymbol{\theta}^{(j+1)} = \tilde{\boldsymbol{\theta}}^{(j+1)}$  if  $u^{(j+1)} < \alpha^{(j+1)}$   
 $= \boldsymbol{\theta}^{(j)}$  if  $u^{(j+1)} \geq \alpha^{(j+1)}$

$$\alpha^{(j+1)} = \min\{1, \exp[\lambda(\boldsymbol{\theta}^{(j+1)}) - \lambda(\boldsymbol{\theta}^{(j)})]\}$$

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(7) Repeat steps (4)-(6) for  
 $j = 2, 3, \dots, 10,000$   
Values of  $\theta^{(j)}$  for  $j \in \{5,001, \dots, 10,000\}$   
represent sample from  $p(\theta|\mathbf{Z})$

choose  $c$  so that have 20-30%  
acceptance at step (6)

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