IV. Markov-switching models

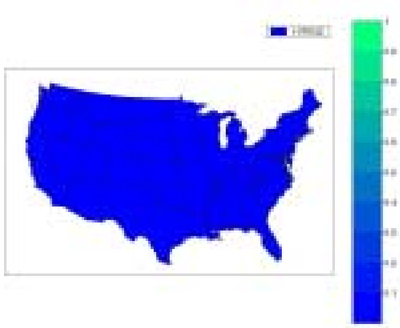
A. Introduction
B. Bayesian analysis of Markov-switching models
C. State-space models with Markov switching
D. Panel models with Markov switching

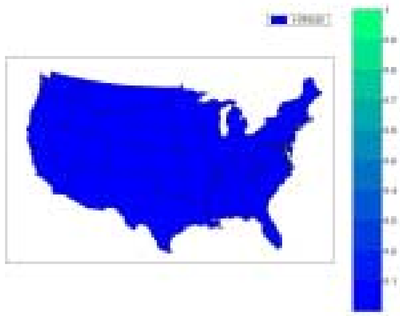
y_{nt} = growth of employment in state n for quarter t

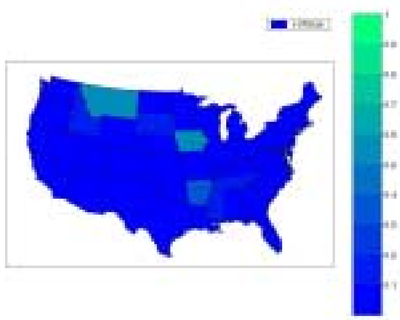
$y_{nt} = \alpha_n + \gamma_n s_{nt} + \varepsilon_{nt}$

$s_{nt} = 1$ when state n is in recession for quarter t

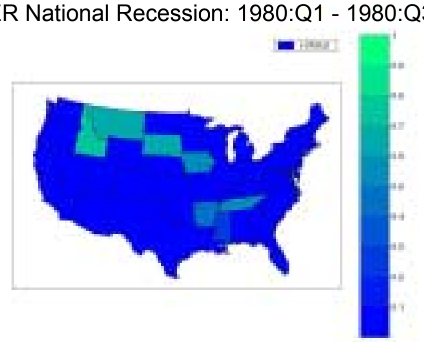
$s_{nt} = 0$ for expansion



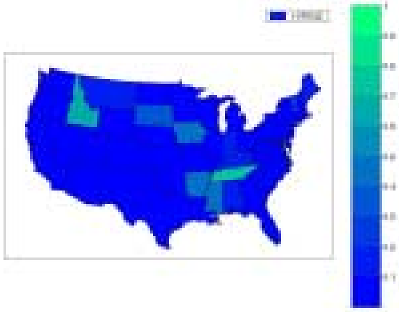




NBER National Recession: 1980:Q1 - 1980:Q3



NBER National Recession: 1980:Q1 - 1980:Q3



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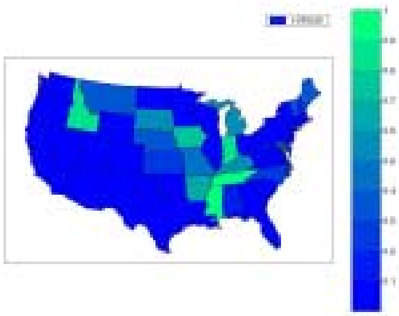

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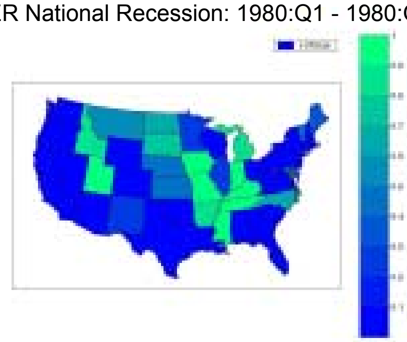

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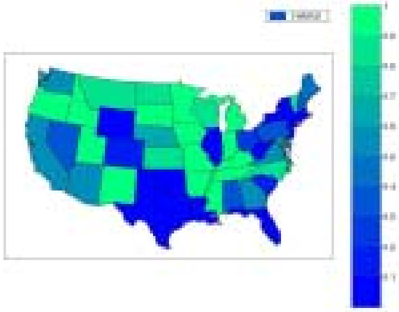

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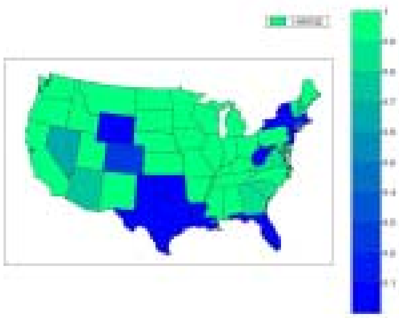

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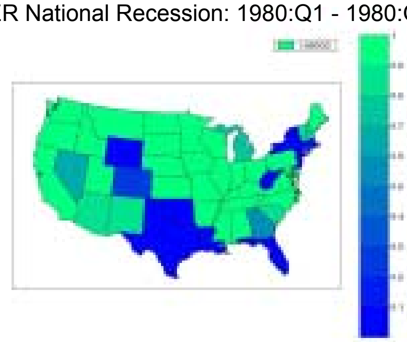

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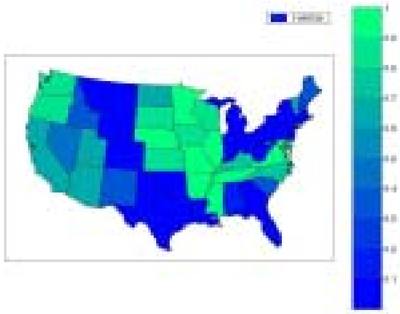
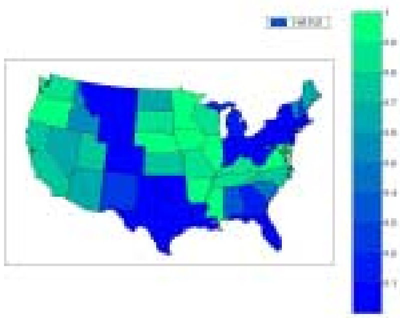
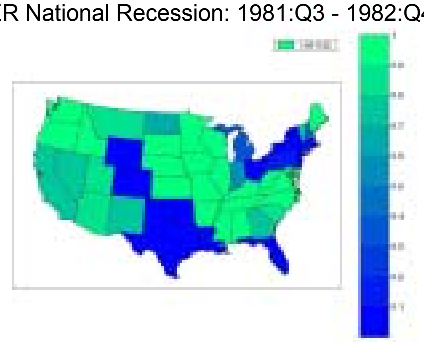
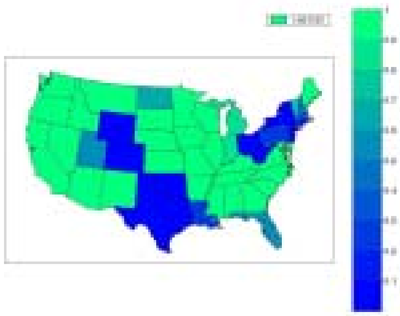




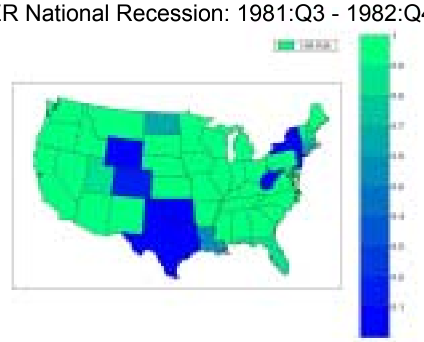
NBER National Recession: 1981:Q3 - 1982:Q4



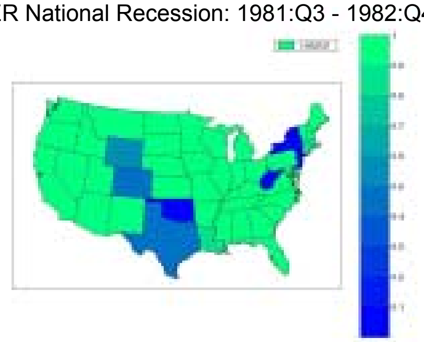
NBER National Recession: 1981:Q3 - 1982:Q4



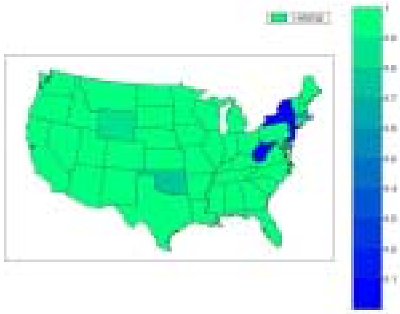
NBER National Recession: 1981:Q3 - 1982:Q4



NBER National Recession: 1981:Q3 - 1982:Q4



NBER National Recession: 1981:Q3 - 1982:Q4



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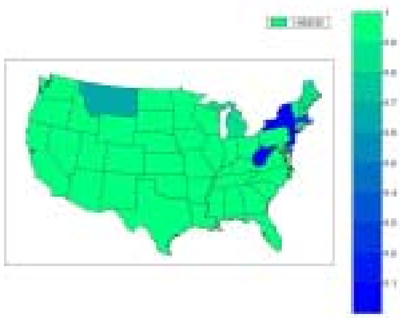
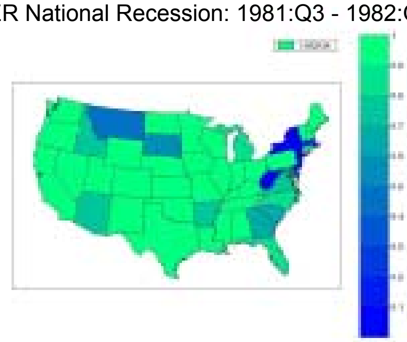
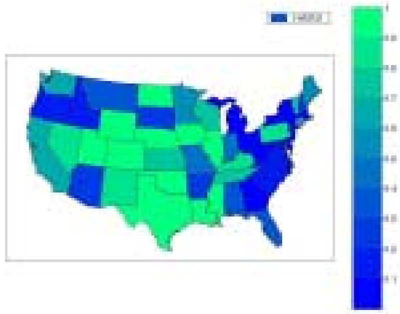


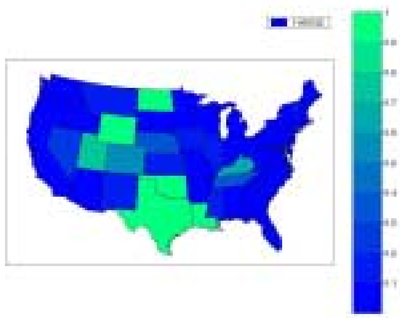
NBER National Recession: 1981:Q3 - 1982:Q4

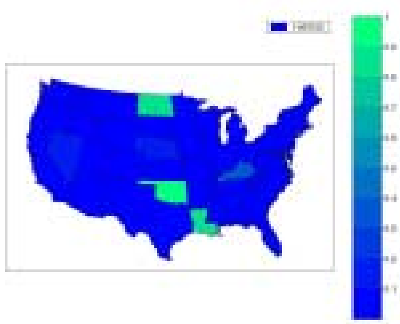


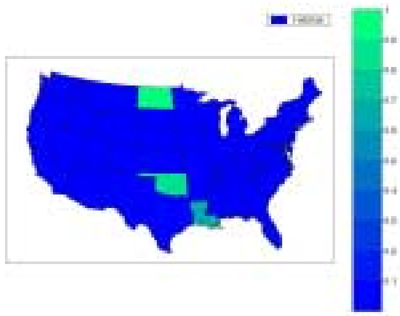
NBER National Recession: 1981:Q3 - 1982:Q4

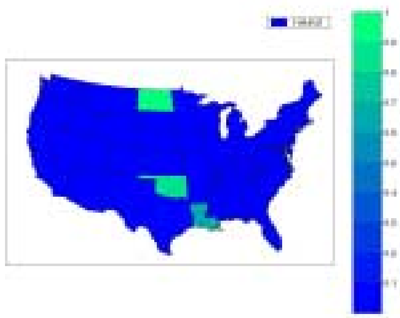


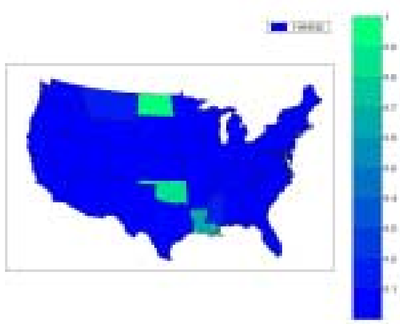
NBER National Recession: 1981:Q3 - 1982:Q4

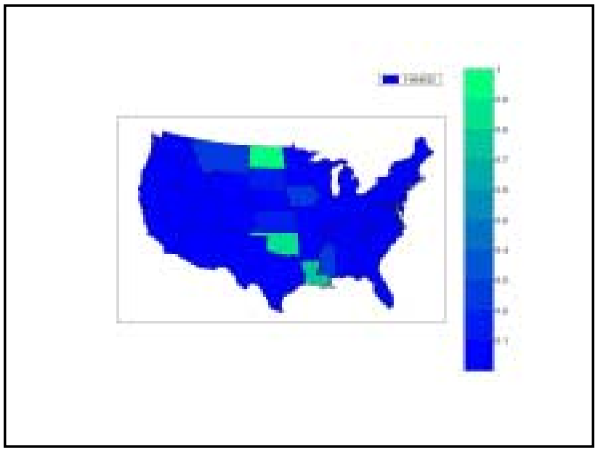


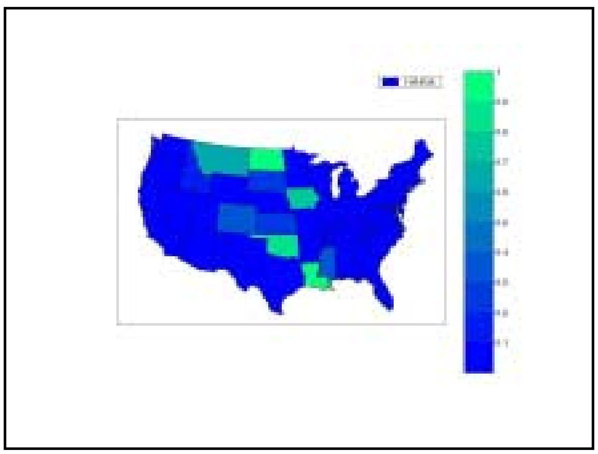


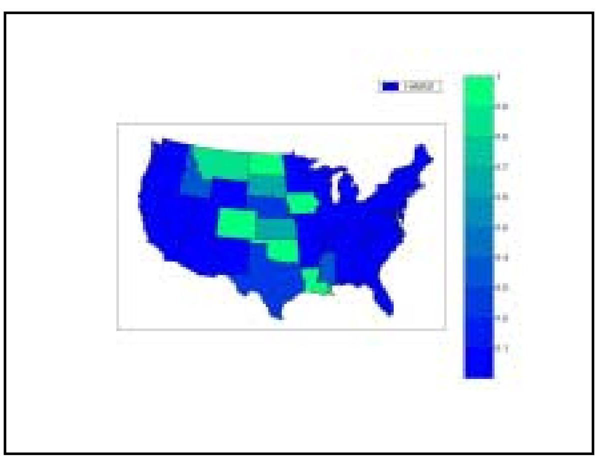


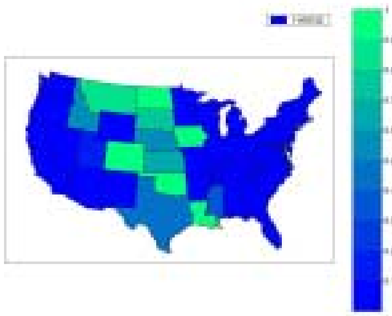


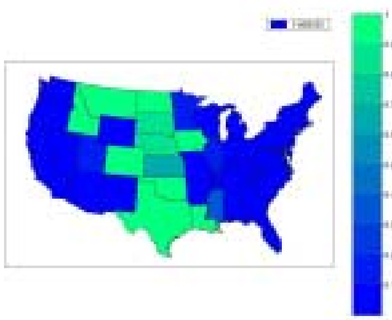


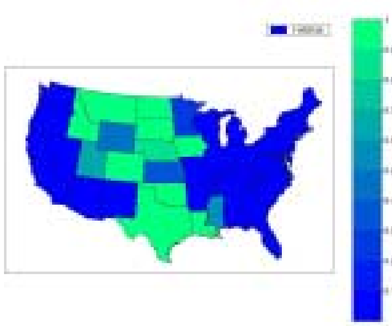


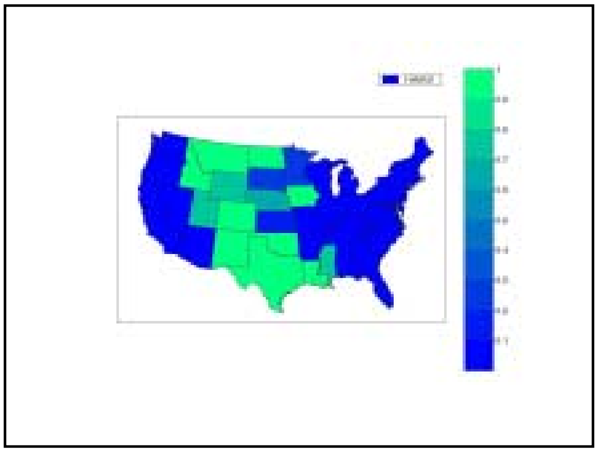


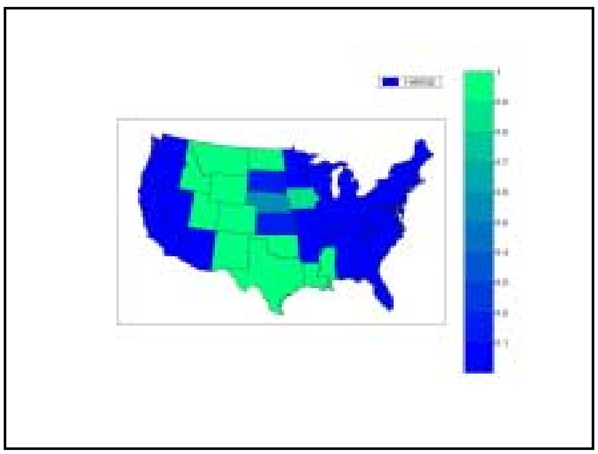


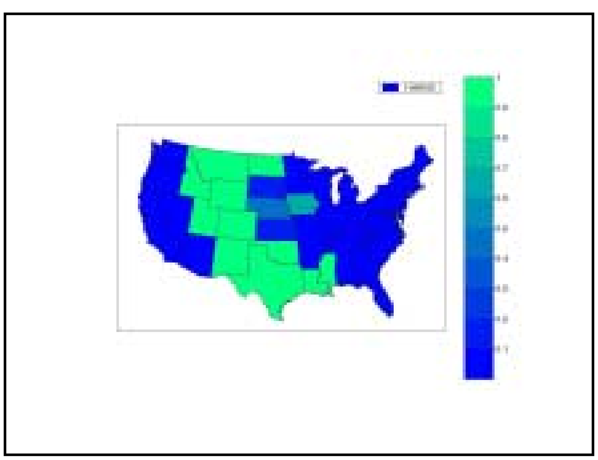


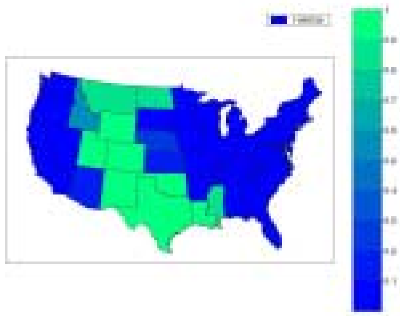


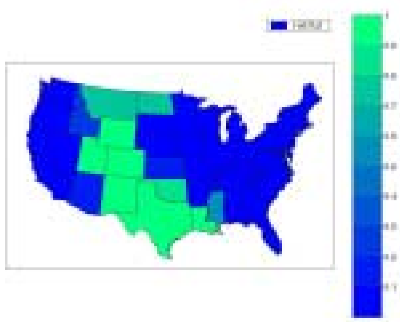


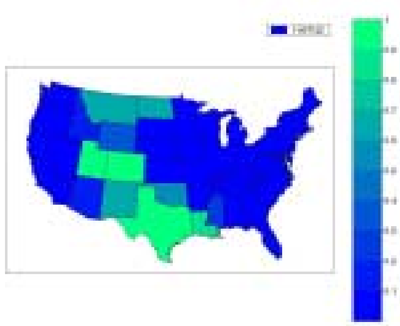












$$\mathbf{y}_t = (y_{1t}, y_{2t}, \ldots, y_{Nt})'$$
$$N = 48$$
$$\mathbf{s}_t = (s_{1t}, s_{2t}, \ldots, s_{Nt})'$$
$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N)'$$
$$\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_N)'$$

$$\mathbf{y}_t = \boldsymbol{\alpha} + \boldsymbol{\gamma} \odot \mathbf{s}_t + \boldsymbol{\varepsilon}_t$$
$$\boldsymbol{\varepsilon}_t | \mathbf{s}_t \sim N(\mathbf{0}, \boldsymbol{\Omega}(\mathbf{s}_t))$$
$$p(\mathbf{s}_t = \mathbf{h}_m | \mathbf{s}_{t-1} = \mathbf{h}_n) = p_{nm}$$

$$\boldsymbol{\Omega}(\mathbf{s}_t) = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_N^2 \end{bmatrix} = \boldsymbol{\Omega}$$

Could in principle use identical algorithm to calculate

$f(\mathbf{y}_1, \ldots, \mathbf{y}_T | \boldsymbol{\alpha}, \boldsymbol{\gamma}, [p_{nm}], \boldsymbol{\Omega}(.))$

Problem:

$[p_{nm}]$ is $(2^N \times 2^N)$

$2^N = 2.8 \times 10^{14}$

Solution: restrict

$\mathbf{s}_t \in \{\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_K\}$

$K << 2^N$

aggregate (unobserved) regime z_t

when $z_t = k$, $\mathbf{s}_t = \mathbf{h}_k$

$\mathbf{y}_t | z_t = k \sim N(\mathbf{m}_k, \boldsymbol{\Omega})$

for $k = 1, \ldots, K$

$\mathbf{m}_k = \boldsymbol{\alpha} + \boldsymbol{\gamma} \odot \mathbf{h}_k$

$\Pr(z_t = j | z_{t-1} = i) = p_{ij}$

for $i, j = 1, \ldots, K$

We impose two of the possible values $(\mathbf{h}_{K-1}, \mathbf{h}_K)$ a priori

$\mathbf{h}_{K-1} = (0, \ldots, 0)'$

national expansion

$\mathbf{h}_K = (1, \ldots, 1)'$

national recession

We think of configuration of each of $(\mathbf{h}_1, \ldots, \mathbf{h}_\kappa)$ for $\kappa = K - 2$ as outcome of unobserved random variables

$\Pr(h_{nk} = 0) = 1/[1 + \exp(\mathbf{x}_n' \boldsymbol{\beta}_k)]$

$\Pr(h_{nk} = 1) = \exp(\mathbf{x}_n' \boldsymbol{\beta}_k)/[1 + \exp(\mathbf{x}_n' \boldsymbol{\beta}_k)]$

$\mathbf{x}_n$ = vector of state characteristics

Unobserved random variables:

$h = \{\mathbf{h}_1, \ldots, \mathbf{h}_\kappa\}$ characterizes which states are in recession when $z_t \in \{1, \ldots, \kappa\}$

$\mathbf{z} = (z_1, \ldots, z_T)'$ characterizes which cluster $k \in \{1, \ldots, K\}$ is in recession at each date

Unknown parameters:

$\Omega = \{\sigma_n\}_{n=1,\ldots,N}$

$\mu = \{\alpha_n, \gamma_n\}_{n=1,\ldots,N}$

$P = \{p_{ij}\}_{i,j=1,\ldots,K}$

$\beta = \{\boldsymbol{\beta}_k\}_{k=1,\ldots,\kappa}$

$\theta = \{\Omega, \mu, P, \beta\}$

$$p(y_{nt}|z_t, h, \theta) \sim N(\alpha_n + \gamma_n s_{nt}(z_t, h), \sigma_n^2)$$

$$p(\mathbf{Y}|\mathbf{z}, h, \theta) = \prod_{t=1}^{T} \prod_{n=1}^{N} f(y_{nt}|z_t, h, \theta)$$

$$p(\mathbf{z}|\theta) = p(z_1) \prod_{t=2}^{T} p(z_t|z_{t-1})$$

$$p(h|\beta) = \prod_{n=1}^{N} \prod_{k=1}^{\kappa} p(h_{nk}|\boldsymbol{\beta}_k)$$

$$p(\mathbf{Y}|\theta) = \sum_{\mathbf{z}} \sum_{h} p(\mathbf{Y}|\mathbf{z}, h, \theta) p(\mathbf{z}|P) p(h|\beta)$$

Classical:

calculating $p(\mathbf{Y}|\boldsymbol{\theta})$ is intractable

Bayesian:

calculating $p(\boldsymbol{\theta}|\mathbf{Y})$ is intractable

Gibbs sampler:

generating a draw from $p(\mathbf{z}|\mathbf{Y}, h, \theta)$, $p(h|\mathbf{Y}, \mathbf{z}, \theta)$, or $p(\theta|\mathbf{Y}, \mathbf{z}, h)$ is fairly simple

Drawing from $\Omega, \mu|\mathbf{Y}, \mathbf{z}, h, P, \beta$ is standard Normal-Gamma regression model

$p(y_{nt}|z_t, h, \theta) \sim N(\alpha_n + \gamma_n s_{nt}(z_t, h), \sigma_n^2)$

Restriction: $\gamma_n < 0$

Drawing from $P|\mathbf{Y}, \mathbf{z}, h, \beta, \mu, \Omega)$

Prior: $p(p_{i1}, \ldots, p_{i,K}) \sim D(\boldsymbol{\alpha})$

$$p(p_{i1}, \ldots, p_{i,K-1}) = \frac{\Gamma(\alpha_1 + \cdots + \alpha_m)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_m)} p_{i1}^{\alpha_1 - 1} \cdots p_{i,K-1}^{\alpha_m - 1}$$

Posterior: $D(\boldsymbol{\alpha}_i^*)$

$$\alpha_{ij}^* = \frac{\sum_{t=2}^{T} \delta(z_{t-1} = i, z_t = j)}{\sum_{t=2}^{T} \delta(z_{t-1} = i)}$$

Drawing from $\mathbf{z}|\mathbf{Y}, h, \theta$
is standard Markov-switching inference

Restrictions:
If $i \leq \kappa$, $j \leq \kappa$, and $i \neq j$
then $p_{ij} = 0$

How to generate posterior logistic?

$$p(h_{nk} = 1) = \frac{\exp(\mathbf{x}_{nk}'\boldsymbol{\beta}_k)}{1 + \exp(\mathbf{x}_{nk}'\boldsymbol{\beta}_k)}$$

We think of this logistic as being generated from two auxiliary variables, as in Andrews-Mallows (1974) and Holmes-Held (2006).

Let $\psi_{nk} \sim$ Kolmogorov-Smirnov

$$p(\psi_{nk}) = 8\sum_{j=1}^{\infty}(-1)^{j+1} j^2 \psi_{nk} \exp(-2j^2\psi_{nk}^2)$$

Let $e_{nk} \sim N(0,1)$

Let $\xi_{nk} = \mathbf{x}'_{nk}\boldsymbol{\beta}_k + 2\psi_{nk} e_{nk}$

Then

$$p(\xi_{nk} > 0) = \frac{\exp(\mathbf{x}'_{nk}\boldsymbol{\beta}_k)}{1 + \exp(\mathbf{x}'_{nk}\boldsymbol{\beta}_k)}$$

$$H_k = \{ \underset{(N\times 1)}{\mathbf{h}_k}, \underset{(N\times 1)}{\boldsymbol{\lambda}_k}, \underset{(N\times 1)}{\boldsymbol{\xi}_k} \}$$

$$H = \{H_j, j = 1, \ldots, \kappa\}$$

$$\kappa = K - 2$$

$$H^{[k]} = \{H_j, j \neq k\}$$

Drawing from $\beta|\mathbf{Y},\mathbf{z},\mu,\Omega,P,H$:

$$\boldsymbol{\xi}_k = \mathbf{X}_k\boldsymbol{\beta}_k + \boldsymbol{\varepsilon}_k$$

$$\boldsymbol{\varepsilon}_k \sim N(\mathbf{0},\mathbf{W}_k)$$

$$\underset{(N\times N)}{\mathbf{W}_k} = diag(\lambda_{1k},\ldots,\lambda_{Nk}).$$

$$\boldsymbol{\beta}_k|\mathbf{Y},\mathbf{z},\mu,\Omega,P,H \sim N(\mathbf{b}_k^*,\mathbf{B}_k^*)$$

Draw from $H_k|H^{[k]},\mathbf{Y},\mathbf{z},\theta$:

(1) $\mathbf{h}_k|H^{[k]},\mathbf{Y},\mathbf{z},\theta$

(2) $\boldsymbol{\xi}_k|\mathbf{h}_k,H^{[k]},\mathbf{Y},\mathbf{z},\theta$

(3) $\boldsymbol{\lambda}_k|\boldsymbol{\xi}_k,\mathbf{h}_k,H^{[k]},\mathbf{Y},\mathbf{z},\theta$

(1) $\mathbf{h}_k|H^{[k]},\mathbf{Y},\mathbf{z},\theta$

$$p(h_{nk}|H^{[k]},\mathbf{Y},\mathbf{z},\theta) = \frac{p(\mathbf{Y}_n|h_{nk},h^{[k]},\theta,\mathbf{z})p(h_{nk}|\boldsymbol{\beta}_k)}{\sum_{j=0}^{1} p(\mathbf{Y}_n|h_{nk}=j,h^{[k]},\theta,\mathbf{z})p(h_{nk}=j|\boldsymbol{\beta}_k)}$$

$$p(h_{nk}=1|\boldsymbol{\beta}) = \frac{\exp(\mathbf{x}_{nk}'\boldsymbol{\beta}_k)}{1+\exp(\mathbf{x}_{nk}'\boldsymbol{\beta}_k)}$$

(2) $\xi_k | \mathbf{h}_k, H^{[k]}, \mathbf{Y}, \mathbf{z}, \theta$

want: Logistic with mean $\mathbf{x}_{nk}'\boldsymbol{\beta}_k$

truncated by $\xi_{nk} \geq 0$ if $h_{nk} = 1$

and $\xi_{nk} < 0$ if $h_{nk} = 0$

If $u \sim U[0,1]$,

and $\xi = A - \log(u^{-1} - 1)$

then $\xi \sim$ Lo(A)

$\xi \geq 0$ iff $u \geq 1/(1 + \exp(A))$

$$u_{nk}^* \sim U[0,1]$$

$$g_{nk} = 1/(1 + \exp(\mathbf{x}_{nk}'\boldsymbol{\beta}_k))$$

$$u_{nk} = g_{nk} u_{nk}^* \text{ if } h_{nk} = 0$$

$$u_{nk} = g_{nk} + g_{nk} \exp(\mathbf{x}_{nk}'\boldsymbol{\beta}_k) u_{nk}^* \text{ if } h_{nk} = 1$$

$$\xi_{nk} = \mathbf{x}_{nk}'\boldsymbol{\beta}_k - \log(u_{nk}^{-1} - 1)$$

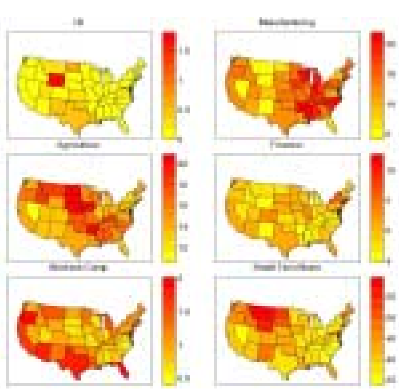
(3) $\lambda_k | \xi_k, \mathbf{h}_k, H^{[k]}, \beta, \theta_1, \theta_2, \theta_3, \mathbf{Y}$

numerically as in Holmes and Held

Empirical results

Explanatory variables for logistic probabilities

- state oil production relative to state GDP
- agricultural employment share
- manufacturing employment share
- financial employment share
- workers compensation
- small firms employment share

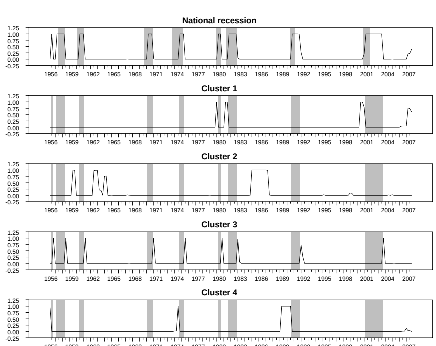


Posterior means of transition probabilities

	from 1	from 2	from 3	from 4	from 5	from 6
to 1	0.56	0	0	0	0.00	0.03
to 2	0	0.77	0	0	0.00	0.03
to 3	0	0	0.00	0	0.24	0.00
to 4	0	0	0	0.63	0.00	0.02
to 5	0.44	0.00	0.00	0.37	0.76	0.03
to 6	0.00	0.23	1.00	0.00	0.00	0.90

	from 1	from 2	from 3	from 4	from 5	from 6
to 1	0.56	0	0	0	0.00	0.03
to 2	0	0.77	0	0	0.00	0.03
to 3	0	0	0.00	0	0.24	0.00
to 4	0	0	0	0.63	0.00	0.02
to 5	0.44	0.00	0.00	0.37	0.76	0.03
to 6	0.00	0.23	1.00	0.00	0.00	0.90

1: early recession
2: separate recession
3: late recovery
4: early recession
5: national recession
6: national expansion



	Cluster 1		Cluster 2		Cluster 3		Cluster 4	
	β_1	δ_1	β_2	δ_2	β_3	δ_3	β_4	δ_4
constant	0.04	---	-0.17	---	-0.10	---	-0.03	---
oil production	-3.3	-0.39	**25.5**	**1.00**	2.7	0.15	-0.6	-0.07
manufactu ring	-0.13	-0.24	**-0.70**	**-0.92**	-0.08	-0.07	0.12	0.21
agriculture	**0.66**	**0.65**	**0.54**	**0.61**	0.23	0.11	**-0.43**	**-0.42**
finance	**-0.47**	**-0.25**	-0.67	-0.40	0.31	0.07	0.06	0.03
workers comp	-0.08	-0.01	-0.52	-0.09	-0.17	-0.01	-0.01	0.00
small firms	-0.11	-0.25	0.06	0.17	**-0.15**	**-0.16**	0.09	0.18