V. Estimation of continuous-time models

$y(t)$ = asset price or interest rate at instant t

$$dy(t) = a[y(t), t; \boldsymbol{\theta}]dt + b[y(t), t; \boldsymbol{\theta}]dW(t)$$

$W(t)$ ~ standard Brownian motion

$$W(t) - W(s) \sim N(0, (t-s))$$

$$y(t+1) = y(t) + \int_{s=t}^{t+1} a[y(s), s; \boldsymbol{\theta}]ds + \int_{s=t}^{t+1} b[y(s), s; \boldsymbol{\theta}]dW(s)$$

Presumptions:

(1) $y(.)$ is only observed at discrete points $t, t+1, t+2, \ldots$

(2) the solution to these integrals is not known analytically

Can approximate using fact that for small Δ,

$$y_{t+\Delta} - y_t \approx N(A_t, B_t)$$
$$A_t = a[y(t), t; \boldsymbol{\theta}]\Delta$$
$$B_t = b^2[y(t), t; \boldsymbol{\theta}]\Delta^2$$

with accuracy arbitrarily good as $\Delta \to 0$

Consider dividing the interval between observations $(t, t+1)$ into M subintervals each of length Δ

$$\tau_{t1} = t + \Delta$$
$$\tau_{t2} = t + 2\Delta$$
$$\vdots$$
$$\tau_{tM} = t + M\Delta$$
$$\tau_{t,M+1} = t + 1$$

with $(M+1)\Delta = 1$

Consider unobserved latent variables
y_{tj}^{*} = "missing" value of $y(\tau_{tj}) = y(t + j\Delta)$
$y_{tj}^{*} \quad j = 1, 2, .., M$ missing between
each pair y_t, y_{t+1} of observed data

$$y_{t,j+1}^{*} - y_{tj}^{*} \sim N(A_{tj}, B_{tj})$$
$$A_{tj} = a[y_{tj}^{*}, \tau_{tj}; \boldsymbol{\theta}]\Delta$$
$$B_{tj} = b^{2}[y_{tj}^{*}, \tau_{tj}; \boldsymbol{\theta}]\Delta^{2}$$

Let $\mathbf{y}_t^{*} = (y_{t1}^{*}, y_{t2}^{*}, \ldots, y_{tM}^{*})'$

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$$p(\mathbf{y}_t^{*} | y_1, y_2, \ldots, y_T, \boldsymbol{\theta})$$
$$= p(\mathbf{y}_t^{*} | \mathbf{y}, \boldsymbol{\theta})$$
$$= \prod_{j=0}^{M} \frac{1}{\sqrt{2\pi B_{tj}}} \exp\left[-\frac{(y_{t,j+1}^{*} - y_{tj}^{*} - A_{tj})^{2}}{2B_{tj}} \right]$$
$$y_{t0}^{*} = y_t$$
$$y_{t,M+1}^{*} = y_{t+1}$$

We can generate a draw from this distribution using Metropolis-Hastings. Let $q(\mathbf{y}_t^*|\mathbf{y},\boldsymbol{\theta})$ be candidate density for iteration k of Gibbs sampler.

(1) generate $\mathbf{w}^{(k)} \sim q(\mathbf{y}_t^*|\mathbf{y},\boldsymbol{\theta})$

(2) calculate $\alpha(\mathbf{y}_t^{*(k-1)},\mathbf{w}^{(k)}|\mathbf{y},\boldsymbol{\theta})$

$$= \min\left\{1, \frac{p\left(\mathbf{w}^{(k)}|\mathbf{y},\boldsymbol{\theta}\right)q\left(\mathbf{y}_t^{*(k-1)}|\mathbf{y};\boldsymbol{\theta}\right)}{p\left(\mathbf{y}_t^{*(k-1)}|\mathbf{y},\boldsymbol{\theta}\right)q\left(\mathbf{w}^{(k)}|\mathbf{y};\boldsymbol{\theta}\right)}\right\}$$

(3) set $\mathbf{y}_t^{*(k)} = \mathbf{w}^{(k)}$ with prob α

set $\mathbf{y}_t^{*(k)} = \mathbf{y}_t^{*(k-1)}$ with prob $1-\alpha$

candidate density:

$$q(\mathbf{y}_t^*|\mathbf{y};\boldsymbol{\theta}) \sim N(\boldsymbol{\mu}^*,\boldsymbol{\Omega}^*)$$

$\boldsymbol{\mu}^*$ = posterior mode

$$= \underset{\mathbf{y}_t^*}{\arg\max}\ p(\mathbf{y}_t^*|\mathbf{y},\boldsymbol{\theta})$$

$$\boldsymbol{\Omega}^* = -\left[\left.\frac{\partial^2 \log p(\mathbf{y}_t^*|\mathbf{y},\boldsymbol{\theta})}{\partial\mathbf{y}_t^*\partial\mathbf{y}_t^{*\prime}}\right|_{\mathbf{y}_t^*=\boldsymbol{\mu}^*}\right]^{-1}$$

Two practical problems:
(1) finding $\boldsymbol{\mu}^*$ requires numerical maximization for each t

Solution: use a small number of Newton-Raphson steps to find approximate $\tilde{\boldsymbol{\mu}}^*$

(2) MH can be very inefficient for drawing a large vector (say, $M = 10$)

Solution(?): generate draw for $y^*_{tj} | \mathbf{y}, y^*_{t1}, y^*_{t2}, \ldots, y^*_{t,j-1}, y^*_{t,j+1}, y^*_{t,j+2}, \ldots, y^*_{tM}, \boldsymbol{\theta})$
But: this won't work because too highly correlated for different j

Solution(?): generate draw for

$y^*_{tj}, y^*_{t,j+1}, \ldots, y^*_{t,j+m} | \mathbf{y}, y^*_{t1}, y^*_{t2}, \ldots, y^*_{t,j-1},$

$y^*_{t,j+m+1}, y^*_{t,j+m+2}, \ldots, y^*_{tM}, \boldsymbol{\theta})$

But: with breaks always in the same place, $\mathbf{y}_t^{*(k)}$ would be too correlated with $\mathbf{y}_t^{*(k+1)}$

Solution(!):

(1) For $k = 1,$ set $\boldsymbol{\theta}^{(k-1)}$ and $\mathbf{y}^{*(k-1)}$ to initial guesses

(2) Set $j = 0$

(3) Draw $m^* \sim$ Poisson(λ)

If $m^* + 1 > M,$ set $m = M$

otherwise, set $m = m^* + 1$

(4) Draw $y^{*(k)}_{tj}, y^{*(k)}_{t,j+1}, \ldots, y^{*(k)}_{t,j+m} | \mathbf{y},$

$\boldsymbol{\theta}, y^*_{ti} = y^{*(k-1)}_{t,i}$

for $i \notin \{j, j+1, \ldots, j+m\}$

and $t = 1, 2, \ldots, T$

(5) Set $j = j + m + 1.$

If $j > M,$ go to (6)

else, go to (3)

(6) Draw $\boldsymbol{\theta}^{(k)}|\mathbf{y}, \mathbf{y}^{*(k)}$
(7) Set $k = k+1$ and go to (2)

How to do step (6)? Conditional on $\mathbf{y}^*$, this is a more standard problem

Example: Cox-Ingersol-Ross model

$$dy(t) = [\alpha - \beta y(t)]dt + \sigma\sqrt{y(t)}\,dW(t)$$

Let $x(t) = \log y(t)$

Ito's Lemma: if

$dy(t) = \mu(y(t),t)dt + \sigma(y(t),t)dW(t)$

$x(t) = f(y(t))$ with $f(.)$ twice continuously differentiable, then

$dx(t) = a(y(t),t)dt + b(y(t),t)dW(t)$

$$a(y(t),t) = \mu(y(t),t)\frac{\partial f(y)}{\partial y}\bigg|_{y=y(t)} +$$
$$(1/2)[\sigma(y(t),t)]^2\frac{\partial^2 f(y)}{\partial y^2}\bigg|_{y=y(t)}$$
$$b(y(t),t) = \sigma(y(t),t)\frac{\partial f(y)}{\partial y}\bigg|_{y=y(t)}$$

Here $f(y) = \log y$

$dy(t) = [\alpha - \beta y(t)]dt + \sigma\sqrt{y(t)}\,dW(t)$

$\partial f/\partial y = 1/y \quad \partial^2 f/\partial y^2 = -1/y^2$

$\sigma(y(t),t) = \sigma\sqrt{y(t)}$

$$b(y(t),t) = \sigma(y(t),t)\frac{\partial f(y)}{\partial y}\bigg|_{y=y(t)}$$
$$= \frac{\sigma\sqrt{y(t)}}{y(t)} = \frac{\sigma}{\sqrt{y(t)}}$$

$$\mu(y(t),t) = \alpha - \beta y(t)$$
$$a(y(t),t) = \mu(y(t),t)\frac{\partial f(y)}{\partial y}\Big|_{y=y(t)} +$$
$$(1/2)[\sigma(y(t),t)]^2 \frac{\partial^2 f(y)}{\partial y^2}\Big|_{y=y(t)}$$
$$= \frac{\alpha}{y(t)} - \beta - \frac{\sigma^2}{2y(t)}$$

$$\log y^*_{tj} = \log y^*_{t,j-1}$$
$$+ \left[\frac{\alpha}{y^*_{t,j-1}} - \beta - \frac{\sigma^2}{2y^*_{t,j-1}} \right]\Delta + u^*_{tj}$$
$$u^*_{tj} \sim N\left(0, \frac{\sigma^2\Delta^2}{y^*_{t,t-1}}\right)$$

goal: rewrite this to recognize as a problem for which we know $p(\alpha, \beta|\sigma^2, \mathbf{y}^*)$ analytically

$$\log y^*_{tj} = \log y^*_{t,j-1}$$
$$+ \left[\frac{\alpha}{y^*_{t,j-1}} - \beta - \frac{\sigma^2}{2y^*_{t,j-1}} \right]\Delta + u^*_{tj}$$
$$u^*_{tj} \sim N\left(0, \frac{\sigma^2\Delta^2}{y^*_{t,t-1}}\right)$$

solution: multiply by $\sqrt{y^*_{t,j-1}}$

$$\tilde{y}_{tj} = \alpha \tilde{x}_{tj} + \beta \tilde{z}_{tj} + u_{tj}$$
$$\tilde{y}_{tj} = \sqrt{y^*_{t,j-1}} \left[\log y^*_{tj} - \log y^*_{t,j-1} + \frac{\sigma^2 \Delta}{y^*_{t,j-1}} \right]$$
$$\tilde{x}_{tj} = \frac{\Delta}{\sqrt{y^*_{t,j-1}}} \qquad \tilde{z}_{tj} = -\Delta \sqrt{y^*_{t,j-1}}$$
$$u_{tj} = \sqrt{y^*_{t,j-1}}\, u^*_{tj} \sim N(0, \sigma^2 \Delta^2)$$

$$\tilde{\mathbf{y}} = (\tilde{y}_{t1}, \tilde{y}_{t2}, \ldots, \tilde{y}_{Tm})'$$
$$\tilde{\mathbf{X}} = \begin{bmatrix} \tilde{x}_{11} & \tilde{z}_{11} \\ \vdots & \vdots \\ \tilde{x}_{TM} & \tilde{z}_{TM} \end{bmatrix}$$
$$\boldsymbol{\beta} = (\alpha, \beta)'$$
$$\tilde{\mathbf{y}} = \tilde{\mathbf{X}}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

prior: $\boldsymbol{\beta}|\sigma^2 \sim N(\mathbf{m}, \sigma\Delta^2\mathbf{M})$

posterior: $\boldsymbol{\beta}|\sigma^{-2}, \tilde{\mathbf{y}}, \tilde{\mathbf{X}} \sim N(\mathbf{m}^*, \sigma^2\Delta^2\mathbf{M}^*)$

$$\mathbf{M}^* = (\mathbf{M}^{-1} + \tilde{\mathbf{X}}'\tilde{\mathbf{X}})^{-1}$$
$$\mathbf{m}^* = \mathbf{M}^*(\mathbf{M}^{-1}\mathbf{m} + \tilde{\mathbf{X}}'\tilde{\mathbf{y}})$$

What about $p(\sigma^2|\alpha, \beta, \mathbf{y}^*)$?
This is not a standard problem, because σ^2 appears both as coefficient and variance.
Chib et.al. draw this numerically