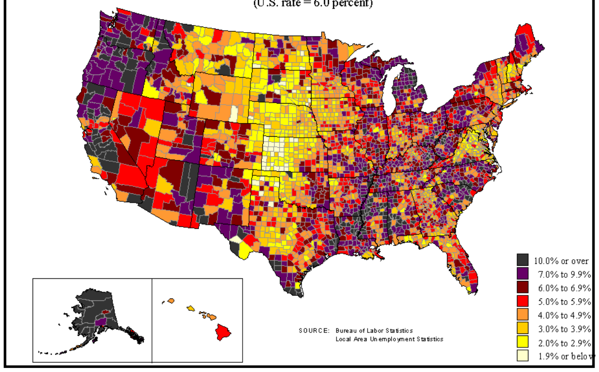
VI. Spatiotemporal models

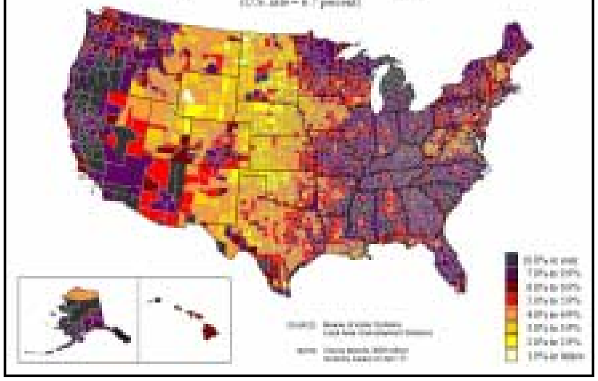
A. Introduction

Unemployment rates by county,
November 2002 - October 2003 averages
(U.S. rate = 6.0 percent)



SOURCE: Bureau of Labor Statistics
Local Area Unemployment Statistics

Unemployment rates by county,
April 2008 – March 2009 averages
(U.S. rate = 6.7 percent)



s = location $s = 1, 2, \ldots, N$

t = date $t = 1, 2, \ldots, T$

$y_t(s)$ = variable of interest

Example 1 (economics):

$s = 1 \Leftrightarrow$ Alabama

$s = N \Leftrightarrow$ Wyoming

$t = 1 \Leftrightarrow$ 1954:I

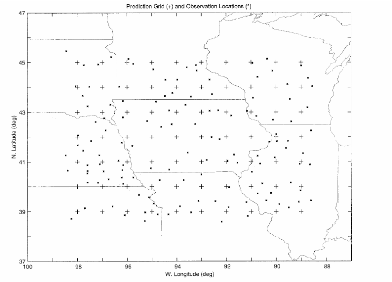
$t = T \Leftrightarrow$ 2004:IV

$y_t(s)$ = unemployment rate in Colorado in 1972:II

Example 2 (Wikle, Berliner, and Cressie):

s = particular location in U.S. midwest at which either temperatures were recorded or are wanted to be inferred

$y_t(s)$ = average daily maximum temperature at location s in month t



$$\mathbf{y}_t = \begin{bmatrix} y_t(1) \\ y_t(2) \\ \vdots \\ y_t(N) \end{bmatrix}$$

$=$ unemployment rates for all states observed in quarter t

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A. Introduction
B. Modeling spatial correlation

time series: when something is observed at one date, it changes what we expect to see at other dates

spatial data: when something is observed at one location, it changes what we expect to see at other locations time series white noise:

$\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$

ε_t independent of ε_τ whenever $t \neq \tau$

---

time series moving average:

$u_t = \varepsilon_t + \theta \varepsilon_{t-1}$

$\Rightarrow u_t$ is correlated with u_{t-1} but not with $u_{t-2}, u_{t-3}, \ldots$

---

time series autoregression:

$u_t = \phi u_{t-1} + \varepsilon_t$

$\Rightarrow u_t$ is correlated with $u_{t-1}, u_{t-2}, \ldots$

spatiotemporal white noise:

$\varepsilon_t(s) \sim N(0, \sigma_\varepsilon^2)$

$\varepsilon_t(s)$ independent of $\varepsilon_\tau(r)$

whenever $t \neq \tau$ or $r \neq s$

spatial moving average:

Let $R(s) =$ set of all states adjacent to s

(note $s \notin R(s)$)

$n(s) =$ number of states adjacent to s

$$u_t(s) = \theta[n(s)]^{-1} \sum_{r \in R(s)} \varepsilon_t(r) + \varepsilon_t(s)$$

$u_t(s)$ is independent of $u_t(r)$ whenever r and s are more than two states apart

Let row s, column r element of $\mathbf{B}$ be $1/n(s)$ if $r \in R(s)$ and zero otherwise

$\mathbf{u}_t = (\mathbf{I}_N + \theta \mathbf{B})\boldsymbol{\varepsilon}_t$

$\mathbf{u}_t \sim N(\mathbf{0}, \sigma^2(\mathbf{I}_N + \theta\mathbf{B})(\mathbf{I}_N + \theta\mathbf{B}'))$

spatial autoregression:

$$u_t(s) = \phi[n(s)]^{-1} \sum_{r\in R(s)} u_t(r) + \varepsilon_t(s)$$

$$\mathbf{u}_t = \phi\mathbf{B}\mathbf{u}_t + \boldsymbol{\varepsilon}_t$$

$$\mathbf{u}_t \sim N(\mathbf{0}, \sigma_\varepsilon^2(\mathbf{I}_N - \phi\mathbf{B})^{-1}(\mathbf{I}_N - \phi\mathbf{B}')^{-1})$$

$u_t(s)$ is correlated with $u_t(r)$ for all r,s

---

suppose there is a shock $a_t \sim N(0,\sigma_a^2)$ that affects all states equally in addition to $u_t(s)$:

$$y_t(s) = a_t + u_t(s)$$

$$\mathbf{y}_t = a_t\mathbf{1}_N + \mathbf{u}_t$$

(where $\mathbf{1}_N$ is the $(N\times 1)$ vector $(1,1,\ldots,1)'$)

$$\mathbf{y}_t \sim N(\mathbf{0}, \sigma_a^2\mathbf{1}_N\mathbf{1}_N' + \sigma_\varepsilon^2(\mathbf{I}_N - \phi\mathbf{B})^{-1}(\mathbf{I}_N - \phi\mathbf{B}')^{-1}))$$

Or we may have prior information about how state s reacts to this shock

$\lambda(s) =$ fraction of workers in state s employed in agriculture

$$y_t = \lambda(s)a_t + u_t(s)$$

$$\mathbf{y}_t \sim N(\mathbf{0}, \sigma_a^2\boldsymbol{\lambda}\boldsymbol{\lambda}' + \sigma_\varepsilon^2(\mathbf{I}_N - \phi\mathbf{B})^{-1}(\mathbf{I}_N - \phi\mathbf{B}')^{-1}))$$

spatial trend:

$j(s)$ = longitude of site s

$k(s)$ = latitude of site s

$q_t(s) = \alpha + \beta j(s) + \gamma k(s)$

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C. Modeling spatiotemporal correlation

$\mathbf{q}_t = (Nx1)$ vector of unobserved factors for each state

$\mathbf{q}_t = \mathbf{H}_1\mathbf{q}_{t-1} + \mathbf{H}_2\mathbf{q}_{t-2} + \cdots + \mathbf{H}_p\mathbf{q}_{t-p} + \boldsymbol{\eta}_t$

Could have spatial correlation structure to $\boldsymbol{\eta}_t$

WBC just take $\boldsymbol{\eta}_t \sim N(\mathbf{0}, \sigma_\eta^2 \mathbf{I}_N)$

and set $p = 1$

$\mathbf{q}_t = \mathbf{H}\mathbf{q}_{t-1} + \boldsymbol{\eta}_t$

We expect that:

$h_{sr} = 0$ if $r \notin R(s)$ and $r \neq s$

WBC impose this outright.

Here we'll use a Bayesian prior that very strongly moves the data towards this without completely forcing:

$h_{sr} | s \neq r, s \notin R(s) \sim N(0, \tau_{h3}^2)$

with τ_{h3}^2 very small

We also might expect h_{ss} to be very similar (but not forced to be identical) for different s

---

Let h_1 be a random variable that summarizes our prior subjective uncertainty about this common value:

$h_1 \sim N(m_{h1}, \tau_{h1}^2)$ where m_{h1} is our prior guess about what h_{ss} would be for a typical state and τ_{h1}^2, our uncertainty about this guess, might be large

Finally, we may expect a pretty similar coefficient (represented by $h_2 \sim N(m_{h2}, \tau_{h2}^2)$) relating $q_t(s)$ to the average of neighboring states' $q_{t-1}(r)$ with τ_{h2}^2 again large

That is, the probability law for our prior on $\mathbf{H}$ can be represented as

$\mathbf{H} = h_1 \mathbf{I}_N + h_2 \mathbf{B} + \mathbf{U}_h$

$h_1 \sim N(m_{h1}, \tau_{h_1}^2)$

$h_2 \sim N(m_{h2}, \tau_{h_2}^2)$

$\mathrm{vec}(\mathbf{U}_h) \sim N(\mathbf{0}, \tau_{h3}^2 \mathbf{I}_{N^2})$

Aside: if we wanted to say there is also a potential aggregate influence of the lagged value of all the other states combined (i.e., of $\mathbf{1}_N' \mathbf{q}_{t-1}$), we could specify

$\mathbf{H} = h_1 \mathbf{I}_N + h_2 \mathbf{B} + h_3 \mathbf{1}_N \mathbf{1}_N' + \mathbf{U}_h$

$\mathbf{H} = h_1\mathbf{I}_N + h_2\mathbf{B} + \mathbf{U}_h$

$\mathbf{i}_N = \mathrm{vec}(\mathbf{I}_N)$

$\mathbf{b} = \mathrm{vec}(\mathbf{B})$

$\mathbf{h} = \mathrm{vec}(\mathbf{H}) = h_1\mathbf{i}_N + h_2\mathbf{b} + \mathrm{vec}(\mathbf{U}_h)$

prior: $\mathbf{h}|\sigma_\eta^2 \sim N(\mathbf{m}_h, \sigma_\eta^2\mathbf{M}_h)$

$\mathbf{m}_h = m_{h1}\mathbf{i}_N + m_{h2}\mathbf{b}$

$\mathbf{M}_h = \tau_{h1}^2\mathbf{i}_N\mathbf{i}_N' + \tau_{h2}^2\mathbf{b}\mathbf{b}' + \tau_{h3}^2\mathbf{I}_{N^2}$

Observed variable to be explained:

$\mathbf{y}_t = (N \times 1)$ vector of unemployment rates for each state

---

Observed explanatory variables:

$\mathbf{w}_t = (k \times 1)$ vector of aggregate explanatory variables (common to all states)

$\mathbf{w}_t = (1, t, \ldots)$

$\mathbf{X}_t = (N \times d)$ matrix of explanatory variables specific for each state (row s = variables for state s)

$$\underset{(N\times1)}{\mathbf{y}_t} = \underset{(N\times k)}{\mathbf{A}}\,\underset{(k\times1)}{\mathbf{w}_t} + \underset{(N\times d)}{\mathbf{X}_t}\,\underset{(d\times1)}{\mathbf{c}} + \underset{(N\times1)}{\mathbf{q}_t}$$

$$+ \underset{(1\times1)}{a_t}\,\underset{(N\times1)}{\mathbf{1}} + \underset{(N\times1)}{\mathbf{u}_t}$$

$$\underset{(N\times1)}{\mathbf{q}_t} = \underset{(N\times N)}{\mathbf{H}}\,\underset{(N\times1)}{\mathbf{q}_{t-1}} + \underset{(N\times1)}{\boldsymbol{\eta}_t}$$

$$a_t = \theta a_{t-1} + v_t$$

$\mathbf{u}_t \sim$ spatial autoregression

$$\mathbf{u}_t = \phi \mathbf{B} \mathbf{u}_t + \boldsymbol{\varepsilon}_t$$

$$\boldsymbol{\eta}_t \sim N(\mathbf{0}, \sigma_\eta^2 \mathbf{I}_N)$$

$$\boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \sigma_\varepsilon^2 \mathbf{I}_N)$$

$$v_t \sim N(0, \sigma_v^2)$$

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State equation:

$$\begin{bmatrix} a_t \\ \mathbf{q}_t \end{bmatrix} = \begin{bmatrix} \theta & \mathbf{0}' \\ \mathbf{0} & \mathbf{H} \end{bmatrix} \begin{bmatrix} a_{t-1} \\ \mathbf{q}_{t-1} \end{bmatrix} + \begin{bmatrix} v_t \\ \boldsymbol{\eta}_t \end{bmatrix}$$

$$\begin{bmatrix} v_t \\ \boldsymbol{\eta}_t \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \sigma_v^2 & \mathbf{0}' \\ \mathbf{0} & \sigma_\eta^2 \mathbf{I}_N \end{bmatrix}\right)$$

Observation equation:

$$\underset{(N\times 1)}{\mathbf{y}_t} = \underset{(N\times k)}{\mathbf{A}}\,\underset{(k\times 1)}{\mathbf{w}_t} + \underset{(N\times d)}{\mathbf{X}_t}\,\underset{(d\times 1)}{\mathbf{c}} + \underset{(N\times 1)}{\mathbf{q}_t} + \underset{(1\times 1)}{a_t}\,\underset{(N\times 1)}{\mathbf{1}} + \underset{(N\times 1)}{\mathbf{u}_t}$$

$$\mathbf{u}_t = \phi \mathbf{B} \mathbf{u}_t + \boldsymbol{\varepsilon}_t$$

$$\boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \sigma_\varepsilon^2 \mathbf{I}_N)$$

Gibbs sampler blocks:

(1) $\boldsymbol{\xi}_t = (a_t, \mathbf{q}_t')'$ for $t = 1, 2, \ldots, T$ posterior distribution known from Kalman filter

Gibbs sampler blocks:

(2) (σ_v^2, θ)

prior:

$\sigma_v^{-2} \sim \Gamma(N_v, \lambda_v)$

$\theta|\sigma_v^2 \sim N(m_\theta, \sigma_v^2 M_\theta)$

data:

$a_t = \theta a_{t-1} + v_t$

$v_t \sim N(0, \sigma_v^2)$

---

posterior:

$\sigma_v^{-2}|\mathbf{a} \sim \Gamma(N_v^*, \lambda_v^*)$

$\theta|\sigma_v^2, \mathbf{a} \sim N(m_\theta^*, \sigma_v^2 M_\theta^*)$

Gibbs sampler blocks:

(3) $(\sigma_\eta^2, \mathbf{H})$

prior:

$\sigma_\eta^{-2} \sim \Gamma(N_\eta, \lambda_\eta)$

$\mathbf{h} = \mathrm{vec}(\mathbf{H})$

$\mathbf{h}|\sigma_\eta^2 \sim N(\mathbf{m}_h, \sigma_\eta^2 \mathbf{M}_h)$

data:

$\mathbf{q}_t = \mathbf{H}\mathbf{q}_{t-1} + \boldsymbol{\eta}_t$

This is a Gaussian VAR, in fact,
even simpler, because lack of correlation
between $\eta_t(s)$ and $\eta_t(r)$ allows this
to be written as a simple univariate
regression.

Notice $\mathbf{Hq}_{t-1} = (\mathbf{q}'_{t-1} \otimes \mathbf{I}_N)\mathbf{h}$

Proof:

$(\mathbf{q}'_{t-1} \otimes \mathbf{I}_N)\mathbf{h} =$

$$\begin{bmatrix} q_{1,t-1}\mathbf{I}_N & q_{2,t-1}\mathbf{I}_N & \cdots & q_{N,t-1}\mathbf{I}_N \end{bmatrix} \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \vdots \\ \mathbf{h}_N \end{bmatrix}$$

$$= q_{1,t-1}\mathbf{h}_1 + q_{2,t-1}\mathbf{h}_2 + \cdots + q_{N,t-1}\mathbf{h}_N$$

$$= \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \cdots & \mathbf{h}_N \end{bmatrix} \begin{bmatrix} q_{1,t-1} \\ q_{2,t-1} \\ \vdots \\ q_{N,t-1} \end{bmatrix}$$

$$= \mathbf{Hq}_{t-1}$$

$$\underset{(N\times N)}{\mathbf{H}}\;\underset{(N\times 1)}{\mathbf{q}_{t-1}} = \left[\underset{(1\times N)}{\mathbf{q}_{t-1}'} \otimes \underset{(N\times N)}{\mathbf{I}_N}\right]\underset{(N^2\times 1)}{\mathbf{h}}$$

$$= \underset{(N\times N^2)}{\tilde{\mathbf{Z}}_t}\;\underset{(N^2\times 1)}{\mathbf{h}}$$

$$\mathbf{q}_t = \mathbf{H}\mathbf{q}_{t-1} + \boldsymbol{\eta}_t$$

$$\mathbf{q}_t = \tilde{\mathbf{Z}}_t\mathbf{h} + \boldsymbol{\eta}_t$$

$$\boldsymbol{\eta}_t \sim N(\mathbf{0}, \sigma_\eta^2\mathbf{I}_N)$$

posterior:

$$\mathbf{h}|\sigma_\eta^2, \mathbf{q} \sim N(\mathbf{m}_h^*, \sigma_\eta^2\mathbf{M}_h^*)$$

$$\mathbf{M}_h^* = \left(\mathbf{M}_h^{-1} + \sum_{t=1}^T \tilde{\mathbf{Z}}_t'\tilde{\mathbf{Z}}_t\right)^{-1}$$

$$\mathbf{m}_h^* = \mathbf{M}_h^*\left(\mathbf{M}_h^{-1}\mathbf{m}_h + \sum_{t=1}^T \tilde{\mathbf{Z}}_t'\mathbf{q}_t\right)$$

$$\sigma_\eta^{-2}|\mathbf{q} \sim \Gamma(N_\eta^*, \lambda_\eta^*)$$

Gibbs sampler blocks:

(4) $(\mathbf{A}, \mathbf{c})$

$$\mathbf{y}_t = \mathbf{A}\mathbf{w}_t + \mathbf{X}_t\mathbf{c} + \mathbf{q}_t + a_t\mathbf{1}_N + \mathbf{u}_t$$

$$\mathbf{u}_t = \phi\mathbf{B}\mathbf{u}_t + \boldsymbol{\varepsilon}_t$$

$$\boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \sigma_\varepsilon^2\mathbf{I}_N)$$

Move $\mathbf{q}_t, a_t$ to LHS and premultiply by $\mathbf{I}_N - \phi\mathbf{B}$:

$$
\begin{aligned}
\mathbf{y}_t &= \mathbf{A}\mathbf{w}_t + \mathbf{X}_t\mathbf{c} + \mathbf{q}_t + a_t\mathbf{1}_N + \mathbf{u}_t \\
\tilde{\mathbf{y}}_t &= \tilde{\mathbf{A}}\mathbf{w}_t + \tilde{\mathbf{X}}_t\mathbf{c} + \boldsymbol{\varepsilon}_t \\
\tilde{\mathbf{y}}_t &= (\mathbf{I}_N - \phi\mathbf{B})(\mathbf{y}_t - \mathbf{q}_t - a_t\mathbf{1}_N) \\
\tilde{\mathbf{A}} &= (\mathbf{I}_N - \phi\mathbf{B})\mathbf{A} \\
\tilde{\mathbf{X}}_t &= (\mathbf{I}_N - \phi\mathbf{B})\mathbf{X}_t
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathbf{y}}_t &= \tilde{\mathbf{A}}\mathbf{w}_t + \tilde{\mathbf{X}}_t\mathbf{c} + \boldsymbol{\varepsilon}_t \\
\boldsymbol{\varepsilon}_t &\sim N(\mathbf{0}, \sigma_\varepsilon^2\mathbf{I}_N) \\
\tilde{\mathbf{A}}\mathbf{w}_t &= (\mathbf{w}_t' \otimes \mathbf{I}_N)\,\mathrm{vec}(\tilde{\mathbf{A}}) \\
\underset{[N\times(Nc+d)]}{\tilde{\mathbf{W}}_t} &= \begin{bmatrix} \underset{(N\times Nc)}{(\mathbf{w}_t' \otimes \mathbf{I}_N)} & \underset{(N\times d)}{\tilde{\mathbf{X}}_t} \end{bmatrix} \\
\underset{[(Nc+d)\times 1]}{\tilde{\boldsymbol{\beta}}} &= \begin{bmatrix} \underset{(Nc\times 1)}{\mathrm{vec}(\tilde{\mathbf{A}})} \\ \underset{(d\times 1)}{\mathbf{c}} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathbf{y}}_t &= \tilde{\mathbf{W}}_t\tilde{\boldsymbol{\beta}} + \boldsymbol{\varepsilon}_t \\
\boldsymbol{\varepsilon}_t &\sim N(\mathbf{0}, \sigma_\varepsilon^2\mathbf{I}_N)
\end{aligned}
$$

prior:

$$\mathbf{a} = \mathrm{vec}(\mathbf{A})$$

$$\mathbf{a}|\sigma_\varepsilon^2,\phi \sim N(\mathbf{m}_A,\sigma^2\mathbf{M}_A)$$

$$\tilde{\mathbf{A}} = (\mathbf{I}_N - \phi\mathbf{B})\mathbf{A}$$

$$\tilde{\mathbf{a}} = \mathrm{vec}(\tilde{\mathbf{A}}) = \mathbf{Q}\mathbf{a}$$

$$\mathbf{Q} = \mathbf{I}_c \otimes (\mathbf{I}_N - \phi\mathbf{B})$$

$$\mathbf{a}|\sigma_\varepsilon^2,\phi \sim N(\mathbf{m}_A,\sigma^2\mathbf{M}_A)$$

$$\tilde{\mathbf{a}} = \mathbf{Q}\mathbf{a}$$

$$\tilde{\mathbf{a}}|\sigma_\varepsilon^2,\phi \sim N(\mathbf{m}_{\tilde{A}},\sigma_\varepsilon^2\mathbf{M}_{\tilde{A}})$$

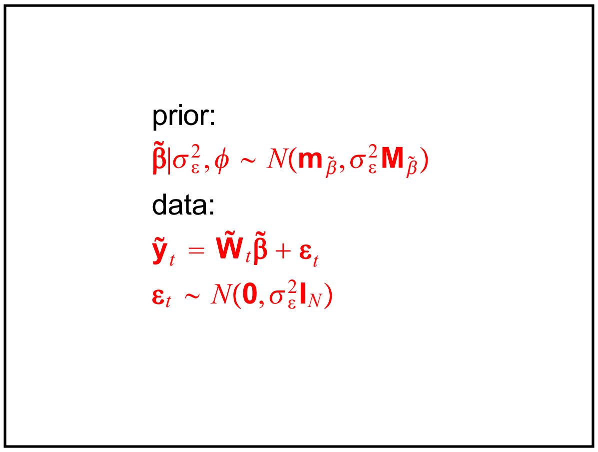
$$\mathbf{m}_{\tilde{A}} = \mathbf{Q}\mathbf{m}_A$$

$$\mathbf{M}_{\tilde{A}} = \mathbf{Q}\mathbf{M}_A\mathbf{Q}'$$

prior:

$$\begin{bmatrix} \tilde{\mathbf{a}}|\sigma_\varepsilon^2,\phi \\ \mathbf{c}|\sigma_\varepsilon^2,\phi \end{bmatrix} \sim N\left(\begin{bmatrix} \mathbf{m}_{\tilde{A}} \\ \mathbf{m}_c \end{bmatrix},\begin{bmatrix} \sigma_\varepsilon^2\mathbf{M}_{\tilde{A}} & \mathbf{0} \\ \mathbf{0} & \sigma_\varepsilon^2\mathbf{M}_c \end{bmatrix}\right)$$

$$\tilde{\boldsymbol{\beta}}|\sigma_\varepsilon^2,\phi \sim N(\mathbf{m}_{\tilde{\beta}},\sigma_\varepsilon^2\mathbf{M}_{\tilde{\beta}})$$



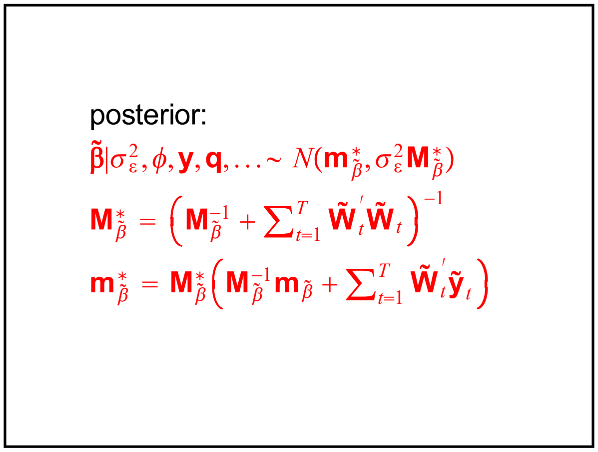
prior:

$$\tilde{\boldsymbol{\beta}}|\sigma_\varepsilon^2,\phi \sim N(\mathbf{m}_{\tilde{\beta}},\sigma_\varepsilon^2\mathbf{M}_{\tilde{\beta}})$$

data:

$$\tilde{\mathbf{y}}_t = \tilde{\mathbf{W}}_t\tilde{\boldsymbol{\beta}} + \boldsymbol{\varepsilon}_t$$

$$\boldsymbol{\varepsilon}_t \sim N(\mathbf{0},\sigma_\varepsilon^2\mathbf{I}_N)$$

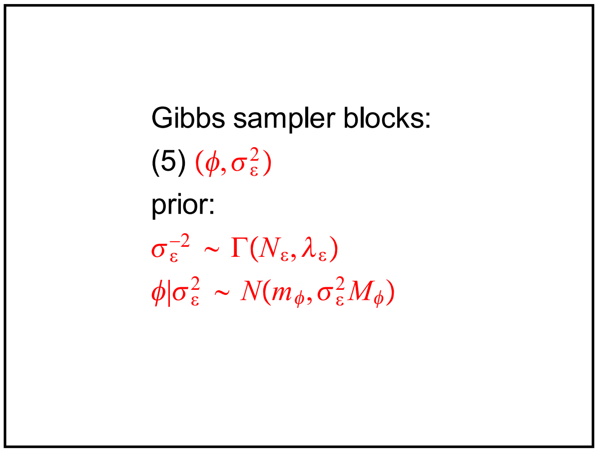


posterior:

$$\tilde{\boldsymbol{\beta}}|\sigma_\varepsilon^2,\phi,\mathbf{y},\mathbf{q},\ldots \sim N(\mathbf{m}_{\tilde{\beta}}^*,\sigma_\varepsilon^2\mathbf{M}_{\tilde{\beta}}^*)$$

$$\mathbf{M}_{\tilde{\beta}}^* = \left(\mathbf{M}_{\tilde{\beta}}^{-1} + \sum_{t=1}^T \tilde{\mathbf{W}}_t'\tilde{\mathbf{W}}_t\right)^{-1}$$

$$\mathbf{m}_{\tilde{\beta}}^* = \mathbf{M}_{\tilde{\beta}}^*\left(\mathbf{M}_{\tilde{\beta}}^{-1}\mathbf{m}_{\tilde{\beta}} + \sum_{t=1}^T \tilde{\mathbf{W}}_t'\tilde{\mathbf{y}}_t\right)$$



Gibbs sampler blocks:

(5) $(\phi,\sigma_\varepsilon^2)$

prior:

$$\sigma_\varepsilon^{-2} \sim \Gamma(N_\varepsilon,\lambda_\varepsilon)$$

$$\phi|\sigma_\varepsilon^2 \sim N(m_\phi,\sigma_\varepsilon^2 M_\phi)$$

data:

$$\mathbf{u}_t = \mathbf{y}_t - \mathbf{A}\mathbf{w}_t - \mathbf{X}_t\mathbf{c} - \mathbf{q}_t - a_t$$

$$\mathbf{u}_t|\phi,\sigma_\varepsilon^2 \sim N(\mathbf{0},\sigma_\varepsilon^2(\mathbf{I}_N - \phi\mathbf{B})^{-1}(\mathbf{I}_N - \phi\mathbf{B}')^{-1})$$

---

posterior:

$$\sigma_\varepsilon^2,\phi|\mathbf{u} \propto |\mathbf{I}_N - \phi\mathbf{B}|^{-T} \times$$

$$\exp\left[-\frac{\sum_{t=1}^{T}(\mathbf{u}_t-\phi\mathbf{B}\mathbf{u}_t)'(\mathbf{u}_t-\phi\mathbf{B}\mathbf{u}_t)}{2\sigma_\varepsilon^2}\right] \times$$

$$(\sigma_\varepsilon^{-2})^{(N_\varepsilon/2)-1}\exp(-\lambda_\varepsilon\sigma_\varepsilon^{-2}/2) \times$$

$$(\sigma_\varepsilon^2)^{-1/2}\exp\left[-\frac{(\phi-m_\phi)^2}{2\sigma_\varepsilon^2 M_\phi}\right]$$

Could generate with Metropolis-Hastings. Candidate density could be the Normal-Gamma posterior if we ignored the Jacobian.

$$
\begin{aligned}
&q(\sigma_\varepsilon^{-2}|\mathbf{u}) \sim \Gamma(N_\varepsilon^*, \lambda_\varepsilon^*)\\
&N_\varepsilon^* = N_\varepsilon + NT\\
&\lambda_\varepsilon^* = \lambda_\varepsilon + \sum_{t=1}^{T} (\mathbf{u}_t - \hat{\phi}\mathbf{B}\mathbf{u}_t)'(\mathbf{u}_t - \hat{\phi}\mathbf{B}\mathbf{u}_t)\\
&\qquad + (\hat{\phi} - m_\phi)^2/Q\\
&\hat{\phi} = S_{xx}^{-1} S_{xy}\\
&S_{xx} = \sum_{t=1}^{T} \mathbf{u}_t'\mathbf{B}'\mathbf{B}\mathbf{u}_t\\
&S_{xy} = \sum_{t=1}^{T} \mathbf{u}_t'\mathbf{B}'\mathbf{u}_t\\
&Q = M_\phi^{-1} S_{xx}/(M_\phi^{-1} + S_{xx})
\end{aligned}
$$

$$
\begin{aligned}
&\phi|\sigma_\varepsilon^2, \mathbf{u} \sim N(m_\phi^*, \sigma_\varepsilon^2 M_\phi^*)\\
&M_\phi^* = (M_\phi^{-1} + S_{xx})^{-1}\\
&m_\phi^* = M_\phi^*(M_\phi^{-1} + S_{xy})
\end{aligned}
$$

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E. Wikle, Berliner, and Cressie results

$$\mathbf{y}_t = \mathbf{A}\mathbf{w}_t + \mathbf{q}_t + \mathbf{u}_t$$
$$\mathbf{w}_t = (1, \cos(2\pi t/12), \sin(2\pi t/12))'$$
$$\mathbf{A} = \begin{bmatrix} \mu_0 & \mathbf{f} & \mathbf{g} \end{bmatrix}$$

spatial trend:

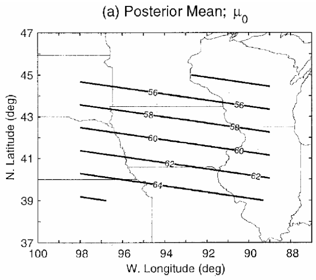
$j(s)$ = longitude of site s

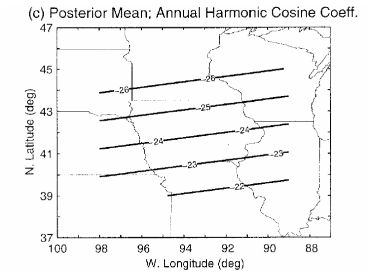
$k(s)$ = latitude of site s

$$\mu_0(s) = \mu_0[1] + \mu_0[2]j(s) + \mu_0[3]k(s)$$
$$f(s) = f[1] + f[2]j(s) + f[3]k(s)$$

priors:

$$\mu_0[r] \sim N(\tilde{\mu}_0[r]\tilde{\sigma}^2_{u_0}[r])$$
$$f[r] \sim N(\tilde{f}[r], \tilde{\sigma}^2_f[r])$$





$$\mathbf{q}_t = \mathbf{H}\mathbf{q}_{t-1} + \boldsymbol{\eta}_t$$

$$h_{ss} = a[j(s),k(s)]$$

$$a(j,k) = a_0(j,k) + \tilde{a}(j,k)$$

$$a_0(j,k) = a_0[1] + a_0[2]j + a_0[3]k$$

$$\tilde{a}(j,k) = \alpha_a[\tilde{a}(j-1,k) + \tilde{a}(j+1,k)]$$

$$+\beta_a[\tilde{a}(j,k-1) + \tilde{a}(j,k+1)]$$