II. Impulse-response functions and variance decompositions

A. Moving average representation and impulse-response function

VAR:

$$\mathbf{y}_t = (n \times 1) \text{ vector}$$

$$\mathbf{y}_t = \mathbf{c} + \Phi_1 \mathbf{y}_{t-1} + \Phi_2 \mathbf{y}_{t-2} + \cdots + \Phi_p \mathbf{y}_{t-p} + \boldsymbol{\varepsilon}_t$$

Can rewrite in "companion form" (e.g., for $p = 3$):

$$\begin{bmatrix} \mathbf{y}_t \\ \mathbf{y}_{t-1} \\ \mathbf{y}_{t-2} \end{bmatrix} = \begin{bmatrix} \mathbf{c} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \Phi_1 & \Phi_2 & \Phi_3 \\ \mathbf{I}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_n & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{y}_{t-1} \\ \mathbf{y}_{t-2} \\ \mathbf{y}_{t-3} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\varepsilon}_t \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

$$\xi_t = \gamma + \mathbf{F}\xi_{t-1} + \mathbf{v}_t$$

$$\xi_t = \begin{bmatrix} \mathbf{y}_t \\ \mathbf{y}_{t-1} \\ \mathbf{y}_{t-2} \end{bmatrix} \qquad \gamma = \begin{bmatrix} \mathbf{c} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} \Phi_1 & \Phi_2 & \Phi_3 \\ \mathbf{I}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_n & \mathbf{0} \end{bmatrix} \qquad \mathbf{v}_t = \begin{bmatrix} \boldsymbol{\varepsilon}_t \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

$$\xi_{t+1} = \gamma + \mathbf{F}\xi_t + \mathbf{v}_{t+1}$$

$$\xi_{t+2} = \gamma + \mathbf{F}\xi_{t+1} + \mathbf{v}_{t+2}$$

$$\xi_{t+2} = (\mathbf{I}_n + \mathbf{F})\gamma + \mathbf{F}^2\xi_t + \mathbf{v}_{t+2} + \mathbf{F}\mathbf{v}_{t+1}$$

$$\xi_{t+s} = (\mathbf{I}_n + \mathbf{F} + \mathbf{F}^2 + \cdots + \mathbf{F}^{s-1})\gamma + \mathbf{F}^s\xi_t + \mathbf{v}_{t+s} + \mathbf{F}\mathbf{v}_{t+s-1} + \mathbf{F}^2\mathbf{v}_{t+s-2} + \cdots + \mathbf{F}^{s-1}\mathbf{v}_{t+1}$$

implication:

$$\underset{(np\times 1)}{\boldsymbol{\xi}_t} = (\mathbf{y}_t', \mathbf{y}_{t-1}', \ldots, \mathbf{y}_{t-p+1}')'$$

$$\frac{\partial \boldsymbol{\xi}_{t+s}}{\partial \boldsymbol{\xi}_t'} = \underset{(np\times np)}{\mathbf{F}^s}$$

$$\frac{\partial \mathbf{y}_{t+s}}{\partial \mathbf{y}_t'} = \underset{(n\times n)}{\boldsymbol{\Psi}_s}$$

$$\begin{aligned}\boldsymbol{\xi}_{t+s} &= (\mathbf{I}_n + \mathbf{F} + \mathbf{F}^2 + \cdots + \mathbf{F}^{s-1})\boldsymbol{\gamma} \\ &\quad + \mathbf{F}^s\boldsymbol{\xi}_t + \mathbf{v}_{t+s} + \mathbf{F}\mathbf{v}_{t+s-1} + \mathbf{F}^2\mathbf{v}_{t+s-2} \\ &\quad + \cdots + \mathbf{F}^{s-1}\mathbf{v}_{t+1} \\ \mathbf{v}_{t+s} &= (\boldsymbol{\varepsilon}_{t+s}', \mathbf{0}', \ldots, \mathbf{0}')'\end{aligned}$$

First n rows:

$$\begin{aligned}\mathbf{y}_{t+s} &= \mathbf{c}_s + \boldsymbol{\varepsilon}_{t+s} + \boldsymbol{\Psi}_1\boldsymbol{\varepsilon}_{t+s-1} + \boldsymbol{\Psi}_2\boldsymbol{\varepsilon}_{t+s-2} + \cdots \\ &\quad + \boldsymbol{\Psi}_{s-1}\boldsymbol{\varepsilon}_{t+1} + \boldsymbol{\Psi}_s\mathbf{y}_t + \mathbf{F}_{12}^{(s)}\mathbf{y}_{t-1} + \\ &\quad + \mathbf{F}_{13}^{(s)}\mathbf{y}_{t-2} + \cdots + \mathbf{F}_{1p}^{(s)}\mathbf{y}_{t-p+1} \\ \mathbf{c}_s &= (\mathbf{I}_n + \boldsymbol{\Psi}_1 + \boldsymbol{\Psi}_2 + \cdots + \boldsymbol{\Psi}_{s-1})\mathbf{c}\end{aligned}$$

$$\underset{(n\times n)}{\boldsymbol{\Psi}_s} = \mathbf{F}_{11}^{(s)} = \frac{\partial \mathbf{y}_{t+s}}{\partial \mathbf{y}_t'} = \frac{\partial \mathbf{y}_{t+s}}{\partial \boldsymbol{\varepsilon}_t'}$$

The plot of $\boldsymbol{\Psi}_s$ as a function of $s = 1, 2, \ldots$ is called the “nonorthogonalized impulse-response function”

Row i, column j of $\boldsymbol{\Psi}_s$ gives effect on $y_{i,t+s}$ of changing y_{jt} (or equivalently, of changing ε_{jt}) holding ε_{kt}, $k \neq j$ and $\boldsymbol{\varepsilon}_{t+m}$, $m \neq 0$ constant

Easier way to calculate:
system is linear, so we can just simulate response of $y_{i,t+s}$ to a unit increase in y_{jt} holding all earlier $y_{k\tau}$ constant

Set $\mathbf{y}_{-1} = \ldots = \mathbf{y}_{-p+1} = \mathbf{0}$

Set $\mathbf{y}_0 = \mathbf{e}_j$ (column j of $\mathbf{I}_n$)

Calculate

$$\mathbf{y}_1 = \mathbf{\Phi}_1\mathbf{y}_0 + \mathbf{\Phi}_2\mathbf{y}_1 + \cdots + \mathbf{\Phi}_p\mathbf{y}_{-p+1}$$

$$\mathbf{y}_2 = \mathbf{\Phi}_1\mathbf{y}_1 + \mathbf{\Phi}_2\mathbf{y}_0 + \cdots + \mathbf{\Phi}_p\mathbf{y}_{-p+2}$$

$$\vdots$$

$$\mathbf{y}_s = \mathbf{\Phi}_1\mathbf{y}_{s-1} + \mathbf{\Phi}_2\mathbf{y}_{s-2} + \cdots + \mathbf{\Phi}_p\mathbf{y}_{s-p}$$

Resulting $\mathbf{y}_s$ is column j of Ψ_s

Repeat for $j = 1, 2, \ldots, n$

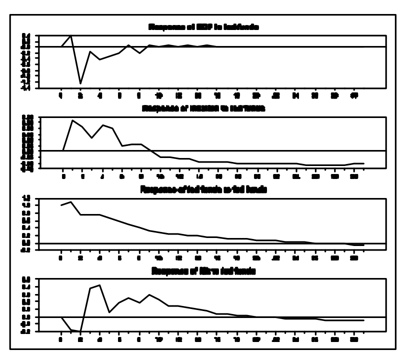
Even easier way to calculate impulse-response function: use RATS command “impulse” (Example 6)

- system(model=basemod)
- variables gdpch inflation fedfunds mgrow
- lags 1 to 5
- det constant
- end(system)

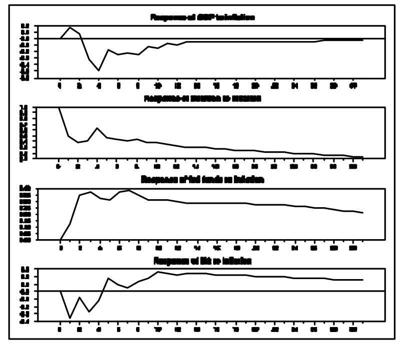
- compute neqn = 4
- compute nsteps = 32

- declare rect[series] impblk(neqn,neqn)
- declare vect[labels] implabel(neqn)
- compute implabel=|| 'GDP', 'inflation', 'fed funds', 'M2' ||
- estimate

- smpl 1 nsteps
- impulse(model=basemod,result=impblk,noprint) neqn nsteps * %identity(neqn)
- do j=1,neqn
- spgraph(vfields=neqn)
- do i=1,neqn
- compute graphlabel='Response of ' +implabel(i) +' to ' +implabel(j)
- graph(header=graphlabel,nodates,number=0) 1
- # impblk(i,j)
- end do j
- spgraph(done)
- end do i



Regression coefficients relating:
GDP(t) to Fed(t-1) = 0.38
Inflation(t) to Fed(t-1) = 0.27
Fed(t) to Fed(t-1) = 1.10
M2(t) to Fed(t-1) = -0.17



$$\underset{(n\times n)}{\boldsymbol{\Psi}_s} = \mathbf{F}_{11}^{(s)} = \frac{\partial \mathbf{y}_{t+s}}{\partial \mathbf{y}_t'} = \frac{\partial \mathbf{y}_{t+s}}{\partial \boldsymbol{\varepsilon}_t'}$$

The plot of Ψ_s as a function of $s = 1,2,\ldots$ is called the "nonorthogonalized impulse-response function"

Note:

$\boldsymbol{\varepsilon}_{t+m}$ for $m \neq 0$ is uncorrelated with ε_{jt} (so this part of "holding constant" makes sense)

ε_{kt} for $k \neq j$ is correlated with ε_{jt} (so this part of "holding constant" may be problematic)

$$\begin{aligned}\boldsymbol{\xi}_{t+s} &= (\mathbf{I}_n + \mathbf{F} + \mathbf{F}^2 + \cdots + \mathbf{F}^{s-1})\boldsymbol{\gamma} \\ &\quad + \mathbf{F}^s\boldsymbol{\xi}_t + \mathbf{v}_{t+s} + \mathbf{F}\mathbf{v}_{t+s-1} + \mathbf{F}^2\mathbf{v}_{t+s-2} \\ &\quad + \cdots + \mathbf{F}^{s-1}\mathbf{v}_{t+1}\end{aligned}$$

If eigenvalues of $\mathbf{F}$ are inside unit circle,

$$\lim_{s\to\infty} \mathbf{F}^s = \mathbf{0}$$

$$\begin{aligned}\boldsymbol{\xi}_{t+s} &= (\mathbf{I}_n - \mathbf{F})^{-1}\boldsymbol{\gamma} + \mathbf{v}_{t+s} + \mathbf{F}\mathbf{v}_{t+s-1} \\ &\quad + \mathbf{F}^2\mathbf{v}_{t+s-2} + \cdots\end{aligned}$$

$$\mathbf{y}_{t+s} = \boldsymbol{\mu} + \boldsymbol{\varepsilon}_{t+s} + \boldsymbol{\Psi}_1\boldsymbol{\varepsilon}_{t+s-1} + \boldsymbol{\Psi}_2\boldsymbol{\varepsilon}_{t+s-2} + \cdots$$

called the MA(∞) representation of the VAR(p)

Note: eigenvalues λ of $\mathbf{F}$ are reciprocals of solutions z to

$$|\mathbf{I}_n - \mathbf{\Phi}_1 z - \mathbf{\Phi}_2 z^2 - \cdots - \mathbf{\Phi}_p z^p| = 0$$

Stationary if

$$\|z\| > 1 \Leftrightarrow \|\lambda\| < 1$$

$$\mathbf{y}_t = \mathbf{c} + \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \mathbf{\Phi}_2 \mathbf{y}_{t-2} + \cdots + \mathbf{\Phi}_p \mathbf{y}_{t-p} + \boldsymbol{\varepsilon}_t$$

II. Impulse-response functions and variance decompositions

A. Moving average representation and impulse-response function

B. Relation between impulse-response function and properties of forecasts

Assumption in Part IA:

$$E(\boldsymbol{\varepsilon}_t \mathbf{y}'_{t-j}) = \mathbf{0} \text{ for } j = 1, 2, \ldots, p$$

(definition of linear projection)

Assumption in Part IIB:

$$E(\boldsymbol{\varepsilon}_t \mathbf{y}'_{t-j}) = \mathbf{0} \text{ for } j = 1, 2, \ldots$$

(p lags capture all serial correlation)

equivalent to:

$$E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}'_{t-j}) = \mathbf{0} \text{ for } j = 1, 2, \ldots$$

We say that "$\boldsymbol{\varepsilon}_t$ is white noise"

$$\mathbf{y}_{t+s} = \mathbf{c}_s + \boldsymbol{\varepsilon}_{t+s} + \mathbf{\Psi}_1 \boldsymbol{\varepsilon}_{t+s-1} + \mathbf{\Psi}_2 \boldsymbol{\varepsilon}_{t+s-2} + \cdots + \mathbf{\Psi}_{s-1} \boldsymbol{\varepsilon}_{t+1} + \mathbf{\Psi}_s \mathbf{y}_t + \mathbf{F}_{12}^{(s)} \mathbf{y}_{t-1} + \mathbf{F}_{13}^{(s)} \mathbf{y}_{t-2} + \cdots + \mathbf{F}_{1p}^{(s)} \mathbf{y}_{t-p+1}$$

Implication of white noise assumption: linear projection of $\mathbf{y}_{t+s}$ on $(1, \mathbf{y}'_t, \mathbf{y}'_{t-1}, \ldots, \mathbf{y}'_{t-p+1})'$ is

$$\hat{\mathbf{y}}_{t+s|t} = \mathbf{c}_s + \mathbf{\Psi}_s \mathbf{y}_t + \mathbf{F}_{12}^{(s)} \mathbf{y}_{t-1} + \mathbf{F}_{13}^{(s)} \mathbf{y}_{t-2} + \cdots + \mathbf{F}_{1p}^{(s)} \mathbf{y}_{t-p+1}$$

Row i of this system is our forecast of $y_{i,t+s}$ based on $\mathbf{y}_t, \mathbf{y}_{t-1}, \ldots$

Row i, column j of $\mathbf{\Psi}_s$ tells us how this forecast changes if we change y_{jt} holding $y_{kt},\ k \neq j$ and $\mathbf{y}_{t-m},\ m > 0$ constant

Equivalently, row i column j of nonorthogonalized impulse-response Ψ_s tells us how our forecast of $y_{i,t+s}$ would change if we already knew all variables dated t and earlier except for y_{jt} and then learn the value of y_{jt}

Another question we could ask: suppose we knew the value of $\mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \ldots$ but only knew one variable for date t (say y_{1t}). How does learning about y_{1t} cause us to change our forecast of $\mathbf{y}_{t+s}$?

Issue: ε_{1t} is correlated with $\varepsilon_{2t}, \ldots, \varepsilon_{nt}$ (through column 1 of Ω). If we learn ε_{1t}, that gives us new information about $\varepsilon_{2t}, \ldots, \varepsilon_{nt}$

By Law of Iterated Projections,

$$\begin{aligned}
&\widehat{E}(\mathbf{y}_{t+s}|1, \varepsilon_{1t}, \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \ldots) \\
&\quad = \mathbf{c}_s + \Psi_s \widehat{E}(\boldsymbol{\varepsilon}_t|\varepsilon_{1t}, \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \ldots) \\
&\quad\quad + \mathbf{F}_{12}^{(s)}\mathbf{y}_{t-1} + \mathbf{F}_{13}^{(s)}\mathbf{y}_{t-2} + \cdots + \mathbf{F}_{1p}^{(s)}\mathbf{y}_{t-p+1}
\end{aligned}$$

$$\begin{aligned}
&\widehat{E}(\mathbf{y}_{t+s}|1, \varepsilon_{1t}, \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \ldots) \\
&\quad = \mathbf{c}_s + \Psi_s \mathbf{a}_1 \varepsilon_{1t} + \mathbf{F}_{12}^{(s)}\mathbf{y}_{t-1} \\
&\quad\quad + \mathbf{F}_{13}^{(s)}\mathbf{y}_{t-2} + \cdots + \mathbf{F}_{1p}^{(s)}\mathbf{y}_{t-p+1}
\end{aligned}$$

$$\mathbf{a}_1 \varepsilon_{1t} = \widehat{E}(\boldsymbol{\varepsilon}_t|\varepsilon_{1t})$$

So, answer to our question, what is effect of y_{1t} alone on forecast of $\mathbf{y}_{t+s}$ if y_{1t} is the only date t variable we have is given by

$$\frac{\partial \widehat{E}(\mathbf{y}_{t+s}|y_{1t}, \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \ldots)}{\partial y_{1t}} = \Psi_s \mathbf{a}_1$$

$$\mathbf{a}_1 = \frac{\partial \widehat{E}(\boldsymbol{\varepsilon}_t|\varepsilon_{1t})}{\partial \varepsilon_{1t}}$$

How to estimate $\mathbf{a}_1$: factor

$$\widehat{\Omega} = T^{-1} \sum_{t=1}^{T} \widehat{\boldsymbol{\varepsilon}}_t \widehat{\boldsymbol{\varepsilon}}_t'$$

as

$$\widehat{\Omega} = \widehat{\mathbf{A}}\widehat{\mathbf{D}}\widehat{\mathbf{A}}'$$

where $\widehat{\mathbf{A}}$ is lower triangular with ones along principal diagonal and $\widehat{\mathbf{D}}$ is diagonal with all diagonal elements positive

$\widehat{\mathbf{a}}_1$ is first column of $\widehat{\mathbf{A}}$

Another way to find $\hat{\mathbf{A}}$ and $\hat{\mathbf{D}}$:

$$\hat{\Omega} = \hat{\mathbf{A}}\hat{\mathbf{D}}\hat{\mathbf{A}}'$$
$$= \hat{\mathbf{A}}\hat{\mathbf{D}}^{1/2}\hat{\mathbf{D}}^{1/2}\hat{\mathbf{A}}'$$
$$= \hat{\mathbf{P}}\hat{\mathbf{P}}'$$

where $\hat{\mathbf{P}}$ is lower triangular with positive elements on principal diagonal ($\hat{\mathbf{P}}$ is Cholesky factor of $\hat{\Omega}$)

So, to find $\hat{\mathbf{A}}$:

(1) Find Cholesky factor $\hat{\Omega} = \hat{\mathbf{P}}\hat{\mathbf{P}}'$

(2) Let $\hat{\mathbf{D}}^{1/2}$ be matrix whose diagonal is diagonal of $\hat{\mathbf{P}}$ and off-diagonal elements are zero

(3) $\hat{\mathbf{A}} = \hat{\mathbf{P}}\hat{\mathbf{D}}^{-1/2}$

$$\frac{\partial \hat{E}(\mathbf{y}_{t+s} | y_{1t}, \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \ldots)}{\partial y_{1t}} = \hat{\Psi}_s \hat{\mathbf{a}}_1$$

Many researchers just report $\hat{\Psi}_s \hat{\mathbf{p}}_1$ where $\hat{\mathbf{p}}_1$ is first column of $\hat{\mathbf{P}}$

Relation between them:

$\hat{\Psi}_s \hat{\mathbf{a}}_1$ is effect on $\mathbf{y}_{t+s}$ of 100-basis-point increase in fed funds

$\hat{\Psi}_s \hat{\mathbf{p}}_1$ is effect of 1-standard-deviation increase in fed funds

$$\hat{\Psi}_s \hat{\mathbf{a}}_1 \hat{p}_{11} = \hat{\Psi}_s \hat{\mathbf{p}}_1$$

Another forecasting question of interest: Suppose I already know $y_{1t}, \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \ldots,$ and now I learn value of y_{2t}. How does this change my forecastof $\mathbf{y}_{t+s}$?

Answer:

$$\frac{\partial \hat{E}(\mathbf{y}_{t+s} | y_{2t}, y_{1t}, \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \ldots)}{\partial y_{2t}} = \hat{\Psi}_s \hat{\mathbf{a}}_2$$

$\hat{\mathbf{a}}_2$ is second column of $\hat{\mathbf{A}}$ where

$$\hat{\Omega} = \hat{\mathbf{A}}\hat{\mathbf{D}}\hat{\mathbf{A}}'$$

Many researchers use $\widehat{\Psi}_s \hat{\mathbf{p}}_2$ for $\hat{\mathbf{p}}_2$ the second column of Cholesky factor of $\widehat{\Omega} = \widehat{\mathbf{P}}\widehat{\mathbf{P}}'$

Definition: the recursively-orthogonalized impulse response function is estimated from $\widehat{\Psi}_s \widehat{\mathbf{A}}$ where $\widehat{\Omega} = \widehat{\mathbf{A}}\widehat{\mathbf{D}}\widehat{\mathbf{A}}'$

Row i, column j element of $\widehat{\Psi}_s \widehat{\mathbf{A}}$ gives

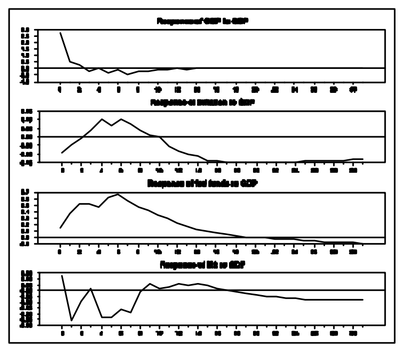
$$\frac{\partial \widehat{E}(y_{i,t+s} | y_{jt}, y_{j-1,t}, \ldots, y_{1t}, \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \ldots)}{\partial y_{jt}}$$

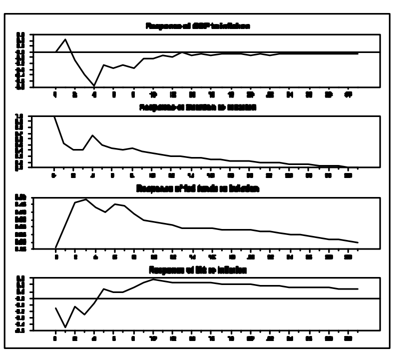
An easier way to calculate Cholesky orthogonalized impulse-response function (Example 7): Replace

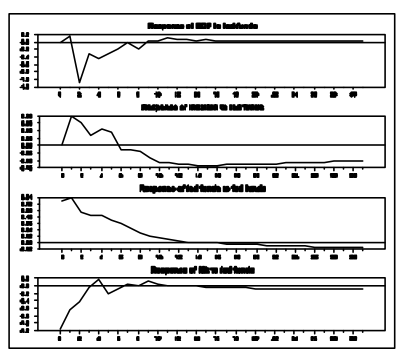
- impulse(model=basemod,result=impblk,noprint) neqn nsteps * %identity(neqn)

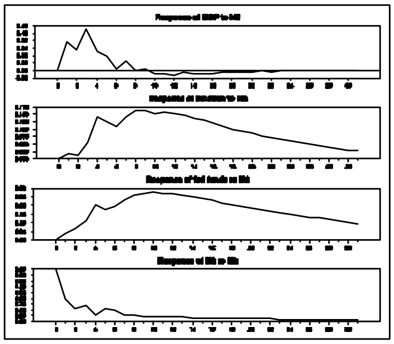
with

- impulse(model=basemod,result=impblk,noprint) neqn nsteps * %sigma









Note: value for recursively-orthogonalized impulse-response function is different for different orderings of variables $y_{1t}, \ldots, y_{nt}$
Which one to use? Depends on which forecasting question you want answer to.

II. Impulse-response functions and variance decompositions

A. Moving average representation and impulse-response function
B. Relation between impulse-response function and properties of forecasts
C. Variance decomposition

$$\mathbf{y}_t = \mathbf{c} + \boldsymbol{\Phi}_1 \mathbf{y}_{t-1} + \boldsymbol{\Phi}_2 \mathbf{y}_{t-2} + \cdots + \boldsymbol{\Phi}_p \mathbf{y}_{t-p} + \boldsymbol{\varepsilon}_t$$

$$E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t') = \boldsymbol{\Omega}$$

$$\boldsymbol{\Omega} = \mathbf{ADA}'$$

A lower triangular, ones on diag
D positive diagonal

Define:

$$\mathbf{u}_t = \mathbf{A}^{-1} \boldsymbol{\varepsilon}_t$$

$$E(\mathbf{u}_t \mathbf{u}_t') = \mathbf{A}^{-1} \boldsymbol{\Omega} (\mathbf{A}^{-1})' = \mathbf{A}^{-1} \mathbf{ADA}' (\mathbf{A}^{-1})' = \mathbf{D} \text{ (diagonal)}$$

$\mathbf{u}_t$ are called “orthogonal innovations”

Interpretation:

$$\boldsymbol{\varepsilon}_t = \mathbf{y}_t - \hat{\mathbf{y}}_{t|t-1}$$

$$u_{1t} = \varepsilon_{1t} = y_{1t} - \widehat{E}(y_{1t} | \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \ldots)$$

$$u_{jt} = y_{jt} - \widehat{E}(y_{jt} | y_{j-1,t}, y_{j-2,t}, \ldots, y_{1t}, \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \ldots) \quad \text{for } j = 2, 3, \ldots, n$$

s-period-ahead forecast error:

$$\mathbf{y}_{t+s} = \mathbf{c}_s + \boldsymbol{\varepsilon}_{t+s} + \boldsymbol{\Psi}_1\boldsymbol{\varepsilon}_{t+s-1} + \boldsymbol{\Psi}_2\boldsymbol{\varepsilon}_{t+s-2} + \cdots + \boldsymbol{\Psi}_{s-1}\boldsymbol{\varepsilon}_{t+1} + \boldsymbol{\Psi}_s\mathbf{y}_t + \mathbf{F}_{12}^{(s)}\mathbf{y}_{t-1} + + \mathbf{F}_{13}^{(s)}\mathbf{y}_{t-2} + \cdots + \mathbf{F}_{1p}^{(s)}\mathbf{y}_{t-p+1}$$

$$\mathbf{y}_{t+s} - \hat{\mathbf{y}}_{t+s|t} = \boldsymbol{\varepsilon}_{t+s} + \boldsymbol{\Psi}_1\boldsymbol{\varepsilon}_{t+s-1} + \boldsymbol{\Psi}_2\boldsymbol{\varepsilon}_{t+s-2} + \cdots + \boldsymbol{\Psi}_{s-1}\boldsymbol{\varepsilon}_{t+1}$$

s-period-ahead mean-squared error:

$$E(\mathbf{y}_{t+s} - \hat{\mathbf{y}}_{t+s|t})(\mathbf{y}_{t+s} - \hat{\mathbf{y}}_{t+s|t})' = \boldsymbol{\Omega} + \boldsymbol{\Psi}_1\boldsymbol{\Omega}\boldsymbol{\Psi}_1' + \boldsymbol{\Psi}_2\boldsymbol{\Omega}\boldsymbol{\Psi}_2' + \cdots + \boldsymbol{\Psi}_{s-1}\boldsymbol{\Omega}\boldsymbol{\Psi}_{s-1}'$$

Can also write in terms of orthogonal innovations:

$$\begin{aligned}\mathbf{y}_{t+s} - \hat{\mathbf{y}}_{t+s|t} &= \boldsymbol{\varepsilon}_{t+s} + \boldsymbol{\Psi}_1\boldsymbol{\varepsilon}_{t+s-1} + \boldsymbol{\Psi}_2\boldsymbol{\varepsilon}_{t+s-2} + \cdots + \boldsymbol{\Psi}_{s-1}\boldsymbol{\varepsilon}_{t+1} \\ &= \mathbf{A}\mathbf{A}^{-1}\boldsymbol{\varepsilon}_{t+s} + \boldsymbol{\Psi}_1\mathbf{A}\mathbf{A}^{-1}\boldsymbol{\varepsilon}_{t+s-1} + \boldsymbol{\Psi}_2\mathbf{A}\mathbf{A}^{-1}\boldsymbol{\varepsilon}_{t+s-2} + \cdots + \boldsymbol{\Psi}_{s-1}\mathbf{A}\mathbf{A}^{-1}\boldsymbol{\varepsilon}_{t+1} \\ &= \mathbf{A}\mathbf{u}_{t+s} + \boldsymbol{\Psi}_1\mathbf{A}\mathbf{u}_{t+s-1} + \boldsymbol{\Psi}_2\mathbf{A}\mathbf{u}_{t+s-2} + \cdots + \boldsymbol{\Psi}_{s-1}\mathbf{A}\mathbf{u}_{t+1}\end{aligned}$$

and s-period-ahead mean squared error can be written

$$E(\mathbf{y}_{t+s} - \hat{\mathbf{y}}_{t+s|t})(\mathbf{y}_{t+s} - \hat{\mathbf{y}}_{t+s|t})' = \mathbf{ADA}' + \boldsymbol{\Psi}_1\mathbf{ADA}'\boldsymbol{\Psi}_1' + \boldsymbol{\Psi}_2\mathbf{ADA}'\boldsymbol{\Psi}_2' + \cdots + \boldsymbol{\Psi}_{s-1}\mathbf{ADA}'\boldsymbol{\Psi}_{s-1}'$$

Can break this down into contribution of each u_{jt}:

$$\mathbf{y}_{t+s} - \hat{\mathbf{y}}_{t+s|t} = \mathbf{A}\mathbf{u}_{t+s} + \boldsymbol{\Psi}_1\mathbf{A}\mathbf{u}_{t+s-1} + \boldsymbol{\Psi}_2\mathbf{A}\mathbf{u}_{t+s-2} + \cdots + \boldsymbol{\Psi}_{s-1}\mathbf{A}\mathbf{u}_{t+1}$$

Write:

$$\mathbf{A}\mathbf{u}_{t+m} = \mathbf{a}_1 u_{1,t+m} + \mathbf{a}_2 u_{2,t+m} + \cdots + \mathbf{a}_n u_{n,t+m}$$

$$\mathbf{A}\mathbf{u}_{t+m} = \mathbf{a}_1 u_{1,t+m} + \mathbf{a}_2 u_{2,t+m} + \cdots + \mathbf{a}_n u_{n,t+m}$$

$$\begin{aligned}E(\mathbf{A}\mathbf{u}_{t+m}\mathbf{u}_{t+m}'\mathbf{A}') &= \mathbf{a}_1\mathbf{a}_1' d_1 + \mathbf{a}_2\mathbf{a}_2' d_2 + \cdots + \mathbf{a}_n\mathbf{a}_n' d_n \\ &= \mathbf{ADA}' = \boldsymbol{\Omega}\end{aligned}$$

s-period-ahead mean squared error:

$$E(\mathbf{y}_{t+s} - \hat{\mathbf{y}}_{t+s|t})(\mathbf{y}_{t+s} - \hat{\mathbf{y}}_{t+s|t})' = \mathbf{ADA}' + \Psi_1 \mathbf{ADA}' \Psi_1' + \Psi_2 \mathbf{ADA}' \Psi_2' + \cdots + \Psi_{s-1} \mathbf{ADA}' \Psi_{s-1}'$$

$$= \sum_{m=0}^{s-1} \Psi_m (\mathbf{a}_1 \mathbf{a}_1' d_1 + \mathbf{a}_2 \mathbf{a}_2' d_2 + \cdots + \mathbf{a}_n \mathbf{a}_n' d_n) \Psi_m'$$

$$E(\mathbf{y}_{t+s} - \hat{\mathbf{y}}_{t+s|t})(\mathbf{y}_{t+s} - \hat{\mathbf{y}}_{t+s|t})' = \sum_{m=0}^{s-1} \Psi_m (\mathbf{a}_1 \mathbf{a}_1' d_1 + \mathbf{a}_2 \mathbf{a}_2' d_2 + \cdots + \mathbf{a}_n \mathbf{a}_n' d_n) \Psi_m'$$

jth diagonal element: mean squared error forecasting $y_{j,t+s}$ at time t

This error results from fact that don't know $\mathbf{u}_{t+1}, \mathbf{u}_{t+2}, \ldots, \mathbf{u}_{t+s}$ at time t

$$E(\mathbf{y}_{t+s} - \hat{\mathbf{y}}_{t+s|t})(\mathbf{y}_{t+s} - \hat{\mathbf{y}}_{t+s|t})' = \sum_{m=0}^{s-1} \Psi_m (\mathbf{a}_1 \mathbf{a}_1' d_1 + \mathbf{a}_2 \mathbf{a}_2' d_2 + \cdots + \mathbf{a}_n \mathbf{a}_n' d_n) \Psi_m'$$

$\sum_{m=0}^{s-1} \Psi_m (\mathbf{a}_k \mathbf{a}_k' d_k) \Psi_m'$ represents how much of this forecast error comes from fact that we don't know $u_{k,t+1}, u_{k,t+2}, \ldots, u_{k,t+s}$ at time t

Tells us how much of s-period-ahead variance comes from $u_{k,t+1}, u_{k,t+2}, \ldots, u_{k,t+s}$

Example 8: variance decomposition

Calculating variance decomposition with RATS: use "errors" in place of "impulse"

- errors(model=basemod,result=impblk) neqn nsteps * %sigma

- Decomposition of Variance for Series GDPCH

Step	Std Error	GDPCH	INFLATION	FEDFUNDS	MGROW
1	2.71891654	100.000	0.000	0.000	0.000
2	2.79124769	97.962	0.630	0.237	1.170
3	3.02670768	84.176	0.801	13.495	1.527
4	3.10543882	80.364	2.284	13.925	3.427
5	3.20084630	75.646	5.563	15.142	3.649
6	3.24924416	74.666	5.921	15.649	3.764
7	3.26938094	73.877	6.599	15.803	3.721
8	3.31708154	74.058	6.876	15.356	3.710
9	3.34176469	73.424	7.519	15.401	3.656
10	3.35393092	73.496	7.563	15.308	3.633
11	3.35885618	73.404	7.697	15.268	3.631
12	3.36337654	73.355	7.713	15.295	3.637

- Decomposition of Variance for Series INFLATION

Step	Std Error	GDPCH	INFLATION	FEDFUNDS	MGROW
1	1.00156337	0.791	99.209	0.000	0.000
2	1.14957413	0.790	96.243	2.941	0.027
3	1.23258222	0.699	95.107	4.160	0.035
4	1.30560394	0.700	95.130	3.975	0.196
5	1.47521569	1.050	94.254	3.655	1.041
6	1.56609260	1.100	93.747	3.600	1.553
7	1.63585173	1.411	93.380	3.322	1.887
8	1.69607437	1.521	92.931	3.108	2.440
9	1.76120133	1.461	92.481	2.936	3.123
10	1.81153093	1.386	91.897	2.982	3.735
11	1.85533966	1.322	91.213	3.227	4.238
12	1.89527654	1.360	90.394	3.487	4.759