III. Structural inference from VAR's

A. Recursive structural models

Consider simple structural model

y_{1t} = real GDP growth

y_{2t} = inflation

y_{3t} = fed funds rate

y_{4t} = rate of growth of M2

(1) aggregate spending responds to shocks only with a 1-quarter delay:

$$y_{1t} = k_1 + \mathbf{b}_1^{(1,.)\prime}\mathbf{y}_{t-1} + \mathbf{b}_2^{(1,.)\prime}\mathbf{y}_{t-2} + \cdots + \mathbf{b}_p^{(1,.)\prime}\mathbf{y}_{t-p} + u_{1t}$$

(2) Phillips curve is relation between current output growth and inflation and lags of all variables:

$$y_{2t} = k_2 + b_0^{(2,1)}y_{1t} + \mathbf{b}_1^{(2,.)\prime}\mathbf{y}_{t-1} + \mathbf{b}_2^{(2,.)\prime}\mathbf{y}_{t-2} + \cdots + \mathbf{b}_p^{(2,.)\prime}\mathbf{y}_{t-p} + u_{2t}$$

(3) Fed sets the funds rate in response to current output and inflation and lags of all variables:

$$y_{3t} = k_3 + b_0^{(3,1)}y_{1t} + b_0^{(3,2)}y_{2t} + \mathbf{b}_1^{(3,.)\prime}\mathbf{y}_{t-1} + \mathbf{b}_2^{(3,.)\prime}\mathbf{y}_{t-2} + \cdots + \mathbf{b}_p^{(3,.)\prime}\mathbf{y}_{t-p} + u_{3t}$$

Key question of interest: if Fed decided to set u_{3t} higher than it usually does, what will be the effects?

$$\partial \mathbf{y}_{t+s}/\partial u_{3t}$$

(4) Money demand depends on current output, inflation, and interest rate:

$$y_{4t} = k_4 + b_0^{(4,1)} y_{1t} + b_0^{(4,2)} y_{2t} + b_0^{(4,3)} y_{3t} + \mathbf{b}_1^{(4,.)\prime} \mathbf{y}_{t-1} + \mathbf{b}_2^{(4,.)\prime} \mathbf{y}_{t-2} + \cdots + \mathbf{b}_p^{(4,.)\prime} \mathbf{y}_{t-p} + u_{4t}$$

Stack these 4 equations into a vector dynamic structural model:

$$\mathbf{B}_0 \mathbf{y}_t = \mathbf{k} + \mathbf{B}_1 \mathbf{y}_{t-1} + \mathbf{B}_2 \mathbf{y}_{t-2} + \cdots + \mathbf{B}_p \mathbf{y}_{t-p} + \mathbf{u}_t$$

where

$$\mathbf{k} = (k_1, k_2, k_3, k_4)'$$

$$\underset{(4\times 4)}{\mathbf{B}_j} = \begin{bmatrix} \mathbf{b}_j^{(1,.)\prime} \\ \mathbf{b}_j^{(2,.)\prime} \\ \mathbf{b}_j^{(3,.)\prime} \\ \mathbf{b}_j^{(4,.)\prime} \end{bmatrix}$$

$$\mathbf{B}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -b_0^{(2,1)} & 1 & 0 & 0 \\ -b_0^{(3,1)} & -b_0^{(3,2)} & 1 & 0 \\ -b_0^{(4,1)} & -b_0^{(4,2)} & -b_0^{(4,3)} & 1 \end{bmatrix}$$

Note we can assume without loss of generality that $\mathbf{u}_t$ is serially uncorrelated. For example, suppose

$$\mathbf{B}_0 \mathbf{y}_t = \mathbf{k} + \mathbf{B}_1 \mathbf{y}_{t-1} + \mathbf{u}_t$$

where $\mathbf{u}_t \sim VAR(1)$

$$\mathbf{u}_t = \mathbf{A}\mathbf{u}_{t-1} + \mathbf{u}_t^*$$

$$\mathbf{B}_0 \mathbf{y}_t - \mathbf{A}\mathbf{B}_0 \mathbf{y}_{t-1} = (\mathbf{I} - \mathbf{A})\mathbf{k} + \mathbf{B}_1 \mathbf{y}_{t-1} - \mathbf{A}\mathbf{B}_1 \mathbf{y}_{t-2} + \mathbf{u}_t^*$$

$$\mathbf{B}_0 \mathbf{y}_t = \mathbf{k}^* + \mathbf{B}_1^* \mathbf{y}_{t-1} + \mathbf{B}_2^* \mathbf{y}_{t-2} + \mathbf{u}_t^*$$

We can also linearize to approximate any dynamic structural model as something of the form

$$\mathbf{B}_0 \mathbf{y}_t = \mathbf{k} + \mathbf{B}_1 \mathbf{y}_{t-1} + \mathbf{B}_2 \mathbf{y}_{t-2} + \cdots + \mathbf{B}_p \mathbf{y}_{t-p} + \mathbf{u}_t$$

But it would be a special model if $\mathbf{B}_0$ had the particular triangular structure posited above

$$\mathbf{B}_0\mathbf{y}_t = \mathbf{k} + \mathbf{B}_1\mathbf{y}_{t-1} + \mathbf{B}_2\mathbf{y}_{t-2} + \cdots + \mathbf{B}_p\mathbf{y}_{t-p} + \mathbf{u}_t$$

But if we premultiply by $\mathbf{B}_0^{-1}$,

$$\mathbf{y}_t = \mathbf{c} + \Phi_1\mathbf{y}_{t-1} + \Phi_2\mathbf{y}_{t-2} + \cdots + \Phi_p\mathbf{y}_{t-p} + \boldsymbol{\varepsilon}_t$$

$$\mathbf{c} = \mathbf{B}_0^{-1}\mathbf{k}$$

$$\Phi_j = \mathbf{B}_0^{-1}\mathbf{B}_j$$

$$\boldsymbol{\varepsilon}_t = \mathbf{B}_0^{-1}\mathbf{u}_t$$

Conclusion: any dynamic structural model of the form

$$\mathbf{B}_0\mathbf{y}_t = \mathbf{k} + \mathbf{B}_1\mathbf{y}_{t-1} + \mathbf{B}_2\mathbf{y}_{t-2} + \cdots + \mathbf{B}_p\mathbf{y}_{t-p} + \mathbf{u}_t$$

with $\mathbf{u}_t$ serially uncorrelated implies a VAR of the form

$$\mathbf{y}_t = \mathbf{c} + \Phi_1\mathbf{y}_{t-1} + \Phi_2\mathbf{y}_{t-2} + \cdots + \Phi_p\mathbf{y}_{t-p} + \boldsymbol{\varepsilon}_t$$

with $\boldsymbol{\varepsilon}_t$ serially uncorrelated

So VAR is just another representation of the structural model

If we knew the structural parameters $(\mathbf{B}_0, \mathbf{B}_1, \ldots)$ and shocks $(\mathbf{u}_t)$ we could calculate the VAR parameters $(\Phi_1, \Phi_2, \ldots)$ and innovations $(\boldsymbol{\varepsilon}_t)$

Any question about the structural model has an analog in an observable property of the VAR, e.g.

$$\frac{\partial \mathbf{y}_{t+m}}{\partial \mathbf{u}_t'} = \frac{\partial \mathbf{y}_{t+m}}{\partial \boldsymbol{\varepsilon}_t'} \frac{\partial \boldsymbol{\varepsilon}_t}{\partial \mathbf{u}_t'} = \Psi_m \mathbf{B}_0^{-1}$$

Does this mean that if we knew the VAR parameters we could infer the structural magnitudes?

Answer: no

VAR parameters:

$$\mathbf{c}, \Phi_1, \Phi_2, \ldots, \Phi_p, E(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t') = n + n^2 p + n(n+1)/2$$

Structural parameters:

$$\mathbf{k}, \mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_p, E(\mathbf{u}_t\mathbf{u}_t') = n + n^2(p+1) + n(n+1)/2$$

Need restrictions, e.g.

$E(\mathbf{u}_t\mathbf{u}_t') =$ diagonal

zeros in $\mathbf{B}_0$

In fact, the earlier triangular model would work perfectly:

$$\mathbf{B}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -b_0^{(2,1)} & 1 & 0 & 0 \\ -b_0^{(3,1)} & -b_0^{(3,2)} & 1 & 0 \\ -b_0^{(4,1)} & -b_0^{(4,2)} & -b_0^{(4,3)} & 1 \end{bmatrix}$$

In a case like this one where there is a one-to-one mapping from VAR parameters to structural parameters, we can find the MLE of the latter from the appropriate transformations of the usual VAR estimates.

Structural model:

$$\mathbf{B}_0\mathbf{y}_t = \mathbf{b}_0 + \mathbf{B}_1\mathbf{y}_{t-1} + \mathbf{B}_2\mathbf{y}_{t-2} + \cdots + \mathbf{B}_p\mathbf{y}_{t-p} + \mathbf{u}_t$$

$E(\mathbf{u}_t\mathbf{u}_t') = \mathbf{D}$ (diagonal)

VAR:

$$\mathbf{y}_t = \mathbf{c} + \Phi_1\mathbf{y}_{t-1} + \Phi_2\mathbf{y}_{t-2} + \cdots + \Phi_p\mathbf{y}_{t-p} + \boldsymbol{\varepsilon}_t$$

$$E(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t') = \boldsymbol{\Omega}$$

$$\boldsymbol{\varepsilon}_t = \mathbf{B}_0^{-1}\mathbf{u}_t$$

$$\boldsymbol{\Omega} = \mathbf{B}_0^{-1}\mathbf{D}(\mathbf{B}_0^{-1})'$$

$$\boldsymbol{\Omega} = \mathbf{B}_0^{-1}\mathbf{D}(\mathbf{B}_0^{-1})'$$

One way to find:

$$\widehat{\boldsymbol{\Omega}} = T^{-1}\sum_{t=1}^{T}\hat{\boldsymbol{\varepsilon}}_t\hat{\boldsymbol{\varepsilon}}_t'$$

If $\mathbf{B}_0$ is lower triangular, so is $\mathbf{B}_0^{-1}$

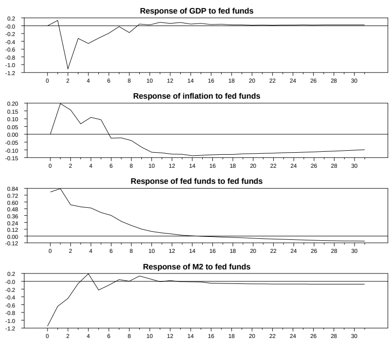
want triangular factorization of $\widehat{\boldsymbol{\Omega}}$

(= Cholesky decomposition)

In other words, the answer to the forecasting question we asked earlier,

$$\frac{\partial E(\mathbf{y}_{t+s}|y_{1t}, y_{2t}, y_{3t}, \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \ldots, \mathbf{y}_{t-p})}{\partial y_{3t}}$$

which we found by using the %sigma specification in the RATS impulse command with fed funds ordered third, is the same as the answer to the structural question $\partial\mathbf{y}_{t+s}/\partial u_{3t}$, which gave the dynamic effects of a shock to monetary policy.

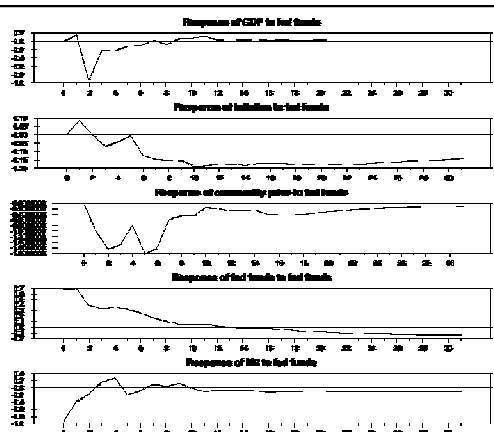


- "Price puzzle": when the Fed tightens, it "causes" higher inflation
- Empirical basis: if the Fed raises funds rate higher than we would expect given current GDP and inflation, it causes us to revise upward our forecast of future inflation

- Plausible interpretation: the Fed raised rates in *anticipation* of higher future inflation
- Possible solution: we need to include in the Fed's behavioral equation the variables on which the Fed is basing its forecast of future inflation
- One common proxy: commodity price index

- *Example 9: Using index of sensitive materials prices to resolve the price puzzle
- open data psm99q_qtr.txt
- data(org=obs) 1948:1 2004:1 date3 ppi
- set ppich = 400*log(ppi(t)/ppi(t-1))

- system(model=basemod)
- variables gdpch inflation ppich fedfunds mgrow
- lags 1 to 5
- det constant
- end(system)



Things we can now calculate:

(1) effect of any structural shock on any variable s periods later:

$$\frac{\partial \mathbf{y}_{t+s}}{\partial \mathbf{u}_t'} = \Psi_s \mathbf{A}$$

Things we can now calculate:

(2) size of any structural shock in any observed period

$$\hat{\mathbf{u}}_t = \hat{\mathbf{B}}_0\hat{\boldsymbol{\varepsilon}}_t = \hat{\mathbf{A}}^{-1}\hat{\boldsymbol{\varepsilon}}_t$$

Things we can now calculate:

(3) Importance of kth structural shock in accounting for s-period-ahead variance of $y_{i,t+s}$

Row i, column i element of

$$\sum_{m=0}^{s-1} \boldsymbol{\Psi}_m(\mathbf{a}_k\mathbf{a}_k' d_k)\boldsymbol{\Psi}_m'$$

The above assumed that commodity prices don't enter the Phillips Curve ($b_0^{(2,3)} = 0$). What if they do?

The above assumed that commodity prices don't enter the Phillips Curve ($b_0^{(2,3)} = 0$). What if they do?

No problem, as long as interest rate does not ($b_0^{(2,4)} = 0$).

$$\mathbf{B}_0 = \begin{bmatrix} x & x & x & 0 & 0 \\ x & x & x & 0 & 0 \\ x & x & x & 0 & 0 \\ x & x & x & x & 0 \\ x & x & x & x & x \end{bmatrix}$$

In this model, the parameters of first three equations and effect of first three shocks is unidentified, but effect of monetary policy shock still is.

Another way to say this: if what we care about is only the effect of variable q, it does not matter how we order variables 1,2,…,q-1 or q+1,q+2,…,n

III. Structural inference from VAR's

A. Recursive structural models
B. Nonrecursive structural models

VAR (reduced-form)

$$\mathbf{y}_t = (n \times 1) \text{ vector}$$

$$\mathbf{y}_t = \mathbf{c} + \Phi_1 \mathbf{y}_{t-1} + \Phi_2 \mathbf{y}_{t-2} + \cdots + \Phi_p \mathbf{y}_{t-p} + \boldsymbol{\varepsilon}_t$$

$$\boldsymbol{\varepsilon}_t = \mathbf{y}_t - \hat{\mathbf{y}}_{t|t-1}$$

$$E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t') = \boldsymbol{\Omega}$$

Structural model:

$$\mathbf{B}_0 \mathbf{y}_t = \mathbf{b}_0 + \mathbf{B}_1 \mathbf{y}_{t-1} + \mathbf{B}_2 \mathbf{y}_{t-2} + \cdots + \mathbf{B}_p \mathbf{y}_{t-p} + \mathbf{u}_t$$

$$E(\mathbf{u}_t \mathbf{u}_{t-m}') = \begin{cases} \mathbf{D} & \text{for } m = 0 \\ \mathbf{0} & \text{for } m \neq 0 \end{cases}$$

Relation between the two: premultiply structural model by $\mathbf{B}_0^{-1}$:

$$\mathbf{B}_0 \mathbf{y}_t = \mathbf{b}_0 + \mathbf{B}_1 \mathbf{y}_{t-1} + \mathbf{B}_2 \mathbf{y}_{t-2} + \cdots + \mathbf{B}_p \mathbf{y}_{t-p} + \mathbf{u}_t$$

$$\mathbf{y}_t = \mathbf{B}_0^{-1} \mathbf{b}_0 + \mathbf{B}_0^{-1} \mathbf{B}_1 \mathbf{y}_{t-1} + \mathbf{B}_0^{-1} \mathbf{B}_2 \mathbf{y}_{t-2} + \cdots + \mathbf{B}_0^{-1} \mathbf{B}_p \mathbf{y}_{t-p} + \mathbf{B}_0^{-1} \mathbf{u}_t$$

$$\mathbf{y}_t = \mathbf{c} + \Phi_1 \mathbf{y}_{t-1} + \Phi_2 \mathbf{y}_{t-2} + \cdots + \Phi_p \mathbf{y}_{t-p} + \boldsymbol{\varepsilon}_t$$

$$\mathbf{y}_t = \mathbf{B}_0^{-1} \mathbf{b}_0 + \mathbf{B}_0^{-1} \mathbf{B}_1 \mathbf{y}_{t-1} + \mathbf{B}_0^{-1} \mathbf{B}_2 \mathbf{y}_{t-2} + \cdots + \mathbf{B}_0^{-1} \mathbf{B}_p \mathbf{y}_{t-p} + \mathbf{B}_0^{-1} \mathbf{u}_t$$

$$\mathbf{y}_t = \mathbf{c} + \Phi_1 \mathbf{y}_{t-1} + \Phi_2 \mathbf{y}_{t-2} + \cdots + \Phi_p \mathbf{y}_{t-p} + \boldsymbol{\varepsilon}_t$$

Can estimate reduced-form by OLS regressions

Goal: put enough restrictions on $\mathbf{D}, \mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_p$ so that there is unique mapping from easily estimated parameters $(\boldsymbol{\Omega}, \Phi_1, \Phi_2, \ldots, \Phi_p)$ into desired parameters $(\mathbf{D}, \mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_p)$

Typical approach:

$\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_p$ unrestricted

$E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t') = \boldsymbol{\Omega}$

$\mathbf{B}_0^{-1} \mathbf{u}_t = \boldsymbol{\varepsilon}_t$

$E(\mathbf{u}_t \mathbf{u}_t') = \mathbf{D}$

$\Rightarrow E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t') = \mathbf{B}_0^{-1} \mathbf{D} (\mathbf{B}_0^{-1})'$

Goal: find $\hat{\mathbf{B}}_0$ and $\hat{\mathbf{D}}$ satisfying restrictions such that

$\hat{\mathbf{B}}_0^{-1} \hat{\mathbf{D}} (\hat{\mathbf{B}}_0^{-1})' = \hat{\boldsymbol{\Omega}}$

$\hat{\boldsymbol{\Omega}} = T^{-1} \sum_{t=1}^{T} \hat{\boldsymbol{\varepsilon}}_t \hat{\boldsymbol{\varepsilon}}_t'$

$\hat{\boldsymbol{\Omega}}$ has $n(n+1)/2$ distinct elements so can have $n(n+1)/2$ unknowns in $\mathbf{B}_0$ and $\mathbf{D}$

One way to find: maximize log likelihood by numerical search

$$\max \; (T/2) \log |\mathbf{B}_0|^2 - (T/2) \log |\mathbf{D}| - (T/2) \operatorname{trace}\left\{ \mathbf{B}_0' \mathbf{D}^{-1} \mathbf{B}_0 \hat{\boldsymbol{\Omega}} \right\}$$

optimal values will satisfy

$\hat{\mathbf{B}}_0^{-1} \hat{\mathbf{D}} (\hat{\mathbf{B}}_0^{-1})' = \hat{\boldsymbol{\Omega}}$

$\boldsymbol{\varepsilon}_t =$ VAR forecast errors

$\mathbf{u}_t =$ true structural disturbances

$\boldsymbol{\varepsilon}_t = \mathbf{B}_0^{-1} \mathbf{u}_t$

$\partial \mathbf{y}_{t+s} / \partial \boldsymbol{\varepsilon}_t' = \boldsymbol{\Psi}_s$ is known

$\partial \mathbf{y}_{t+s} / \partial \mathbf{u}_t'$ is desired

$\partial \mathbf{y}_{t+s} / \partial \mathbf{u}_t' = \boldsymbol{\Psi}_s \mathbf{B}_0^{-1}$

- Implement with RATS cvmodel command
- Example 10: Implementing the recursive Cholesky factorization by hand

- system(model= basemod)
- variables gdpch inflation ppich fedfunds mgrow
- lags 1 to 5
- det constant
- end(system)
- estimate

- system(model= basemod)
- variables gdpch inflation ppich fedfunds mgrow
- lags 1 to 5
- det constant
- end(system)
- estimate

%sigma is now VAR residual variance-covariance $\hat{\mathbf{\Omega}} = \mathbf{T}^{-1}\sum_{t=1}^{T}\hat{\boldsymbol{\varepsilon}}_t\hat{\boldsymbol{\varepsilon}}_t'$

- nonlin a21 a31 a32 a41 a42 a43 a51 a52 a53 a54
- dec frml[rect] afrml
- frml afrml = || 1.0, 0.0, 0.0, 0.0, 0.0 | $
- a21, 1.0, 0.0, 0.0, 0.0 | $
- a31, a32, 1.0, 0.0, 0.0 | $
- a41, a42, a43, 1.0, 0.0 | $
- a51, a52, a53, a54, 1.0 ||
- compute a21 = a31 = a32 = a41 = a42 = a43 = a51 = a52 = a53 = a54 = 0
- cvmodel(iters=100,method=bfgs,trace, factor=afactor) %sigma afrml

---

- nonlin a21 a31 a32 a41 a42 a43 a51 a52 a53 a54
- dec frml[rect] afrml
- frml afrml = || 1.0, 0.0, 0.0, 0.0, 0.0 | $
- a21, 1.0, 0.0, 0.0, 0.0 | $
- a31, a32, 1.0, 0.0, 0.0 | $
- a41, a42, a43, 1.0, 0.0 | $
- a51, a52, a53, a54, 1.0 ||
- compute a21 = a31 = a32 = a41 = a42 = a43 = a51 = a52 = a53 = a54 = 0
- cvmodel(iters=100,method=bfgs,trace, factor=afactor) %sigma afrml

a21, a31, ... are now values that maximize
$(T/2)\log|\mathbf{B}_0|^2 - (T/2)\log|\mathbf{D}| - (T/2)\,\mathrm{trace}\{\mathbf{B}_0'\mathbf{D}^{-1}\mathbf{B}_0\hat{\mathbf{\Omega}}\}$

- compute aest = || 1.0, 0.0, 0.0, 0.0, 0.0 | $
- a21, 1.0, 0.0, 0.0, 0.0 | $
- a31, a32, 1.0, 0.0, 0.0 | $
- a41, a42, a43, 1.0, 0.0 | $
- a51, a52, a53, a54, 1.0 ||
- compute aest = inv(aest)

- compute aest = || 1.0, 0.0, 0.0, 0.0, 0.0 | $
- a21, 1.0, 0.0, 0.0, 0.0 | $
- a31, a32, 1.0, 0.0, 0.0 | $
- a41, a42, a43, 1.0, 0.0 | $
- a51, a52, a53, a54, 1.0 ||
- compute aest = inv(aest)

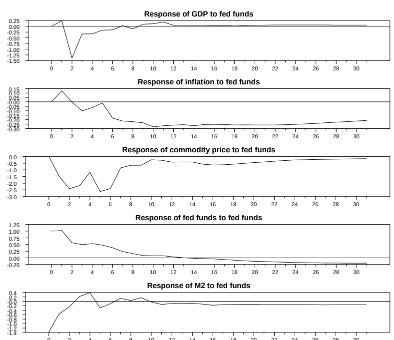
aest is now $\hat{\mathbf{B}}_0^{-1}$

Plots of $\partial\mathbf{y}_{t+s}/\partial\mathbf{u}_t' = \hat{\mathbf{\Psi}}_s\hat{\mathbf{B}}_0^{-1}$ are now obtained as follows:

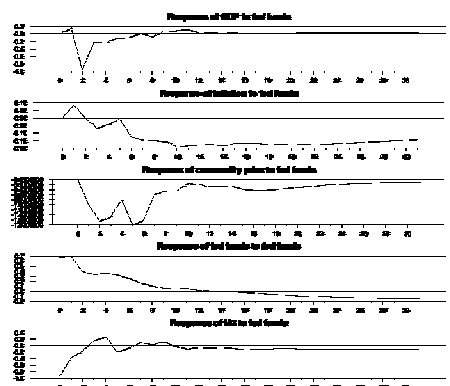
- compute neqn = 5
- compute nsteps = 32
- declare vect[labels] implabel(neqn)
- compute implabel=|| 'GDP', 'inflation', 'commodity price', 'fed funds', 'M2' ||
- declare rect[series] impblk(neqn,1)

- smpl 1 nsteps
- do j=1,neqn
- impulse(model=basemod, shock=%xcol(aest,j), result=impblk) neqn nsteps
- spgraph(vfields=neqn)
- do i=1,neqn
- compute graphlabel='Response of ' +implabel(i) +' to ' +implabel(j)
- graph(header=graphlabel,nodates,number=0) 1
- # impblk(i,1)
- end do i
- spgraph(done)
- end do j

Example 10: IRF from recursive nonlinear estimation



Example 9: IRF from Cholesky factorization

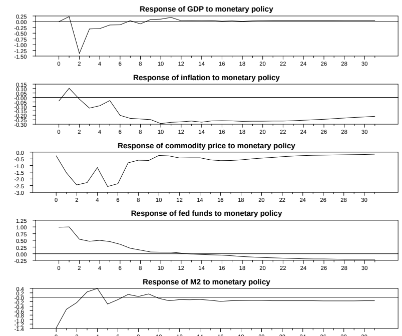


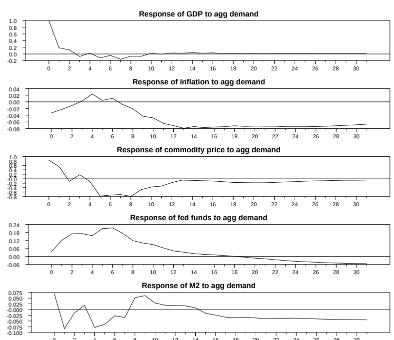
Example 11: nonrecursive structure

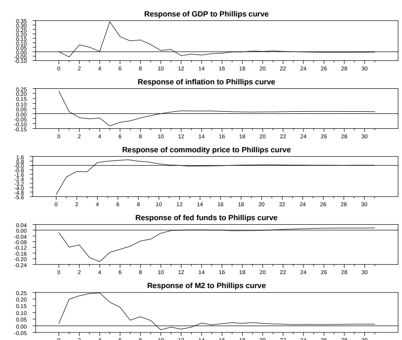
(1) Aggregate demand responds to variables only with a lag
(2) Phillips curve includes commodity price (inflationary expectations)
(3) Inflation expectations respond to current inflation and money growth
(4) Fed responds to output, inflation, and commodity price
(5) Money demand does not depend on commodity price

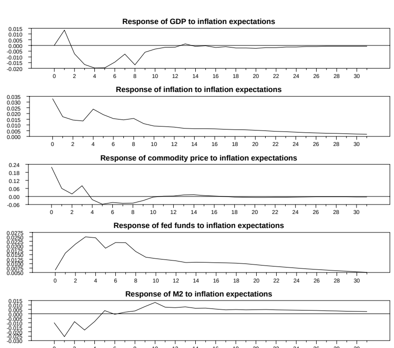
	GDP	inflation	com price	fed funds	M2
aggregate demand	x	0	0	0	0
Phillips curve	x	x	x	0	0
inflation expectations	0	x	x	0	x
monetary policy	x	x	x	x	0
money demand	x	x	0	x	x

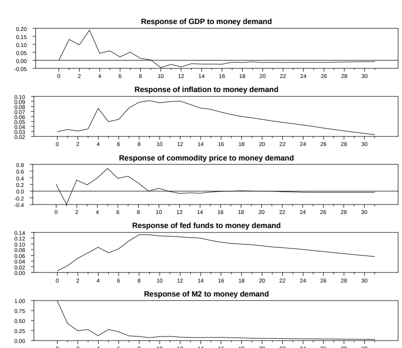
- nonlin a21 a23 a32 a35 a41 a42 a43 a51 a52 a54
- dec frml[rect] afrml
- frml afrml = || 1.0, 0.0, 0.0, 0.0, 0.0 | $
- a21, 1.0, a23, 0.0, 0.0 | $
- 0.0, a32, 1.0, 0.0, a35 | $
- a41, a42, a43, 1.0, 0.0 | $
- a51, a52, 0, a54, 1.0 ||











III. Structural inference from VAR's

A. Recursive structural models
B. Nonrecursive structural models
C. Identification using long-run restrictions

x log of productivity
(log GDP minus log civilian labor force)
n log of civilian labor force

$$\mathbf{y}_t = \begin{bmatrix} \Delta x_t \\ \Delta n_t \end{bmatrix} \sim I(0)$$

VAR (reduced-form)

$$\mathbf{y}_t = \mathbf{c} + \boldsymbol{\Phi}_1 \mathbf{y}_{t-1} + \boldsymbol{\Phi}_2 \mathbf{y}_{t-2} + \cdots + \boldsymbol{\Phi}_p \mathbf{y}_{t-p} + \boldsymbol{\varepsilon}_t$$

$$E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t') = \boldsymbol{\Omega}$$

Structural model:

$$\mathbf{B}_0 \mathbf{y}_t = \mathbf{b}_0 + \mathbf{B}_1 \mathbf{y}_{t-1} + \mathbf{B}_2 \mathbf{y}_{t-2} + \cdots + \mathbf{B}_p \mathbf{y}_{t-p} + \mathbf{u}_t$$

$E(\mathbf{u}_t \mathbf{u}_t') = \mathbf{I}_2$ (normalization)

Relation between representations:

$$\mathbf{u}_t = \mathbf{B}_0 \boldsymbol{\varepsilon}_t$$

$$\boldsymbol{\Omega} = \mathbf{B}_0^{-1} (\mathbf{B}_0^{-1})'$$

Premultiply structural model,

$$\mathbf{B}(L)\mathbf{y}_t = \mathbf{b}_0 + \mathbf{u}_t$$

by $\mathbf{C}(L) = \mathbf{B}(L)^{-1}$:

$$\mathbf{y}_t = \boldsymbol{\mu} + \mathbf{C}_0\mathbf{u}_t + \mathbf{C}_1\mathbf{u}_{t-1} + \mathbf{C}_2\mathbf{u}_{t-2} + \cdots$$

which gives structural MA representation

$$\mathbf{u}_t = \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix}$$

u_{1t} technology shock

u_{2t} demand disturbances

Assumption: demand shocks can not have a permanent effect on productivity

$$\lim_{s\to\infty} \frac{\partial x_{t+s}}{\partial u_{2t}} = 0$$

Notice

$$\frac{\partial x_{t+s}}{\partial u_{2t}} = \frac{\partial(x_{t+s} - x_{t+s-1})}{\partial u_{2t}} + \frac{\partial(x_{t+s-1} - x_{t+s-2})}{\partial u_{2t}} + \cdots + \frac{\partial(x_t - x_{t-1})}{\partial u_{2t}}$$

$$\mathbf{y}_t = \begin{bmatrix} x_t - x_{t-1} \\ n_t - n_{t-1} \end{bmatrix}$$

$$\frac{\partial(x_t - x_{t-1})}{\partial u_{2t}} = \frac{\partial y_{1t}}{\partial u_{2t}}$$

$$\mathbf{y}_t = \boldsymbol{\mu} + \mathbf{C}_0\mathbf{u}_t + \mathbf{C}_1\mathbf{u}_{t-1} + \mathbf{C}_2\mathbf{u}_{t-2} + \cdots$$

$$\frac{\partial \mathbf{y}_{t+m}}{\partial \mathbf{u}_t'} = \mathbf{C}_m$$

$$\frac{\partial x_{t+s}}{\partial u_{2t}} = \frac{\partial(x_{t+s} - x_{t+s-1})}{\partial u_{2t}} + \frac{\partial(x_{t+s-1} - x_{t+s-2})}{\partial u_{2t}} + \cdots + \frac{\partial(x_t - x_{t-1})}{\partial u_{2t}}$$

is given by the row 1 column 2 element of

$$\mathbf{C}_0 + \mathbf{C}_1 + \mathbf{C}_2 + \cdots + \mathbf{C}_s$$

$$\lim_{s\to\infty} \frac{\partial x_{t+s}}{\partial u_{2t}} = 0$$

requires that the following matrix is lower triangular:

$$\mathbf{C}_0 + \mathbf{C}_1 + \mathbf{C}_2 + \cdots = \mathbf{C}(1)$$

Goal: find structural disturbances $\mathbf{u}_t$ that are a linear combination of the VAR innovations,

$$\mathbf{u}_t = \mathbf{H}\boldsymbol{\varepsilon}_t,$$

such that:

(1) $E(\mathbf{u}_t\mathbf{u}_t') = \mathbf{I}_2$

$$\Rightarrow \mathbf{H}\boldsymbol{\Omega}\mathbf{H}' = \mathbf{I}_2$$

$$\Rightarrow \boldsymbol{\Omega} = (\mathbf{H}^{-1})(\mathbf{H}^{-1})'$$

(2) $\mathbf{y}_t = \boldsymbol{\mu} + \mathbf{C}(L)\mathbf{u}_t$

(3) $\mathbf{C}(1)$ is lower triangular

$$\boldsymbol{\Phi}(L)\mathbf{y}_t = \mathbf{c} + \boldsymbol{\varepsilon}_t$$

$$\boldsymbol{\varepsilon}_t = \mathbf{H}^{-1}\mathbf{u}_t$$

$$\Rightarrow \boldsymbol{\Phi}(L)\mathbf{y}_t = \mathbf{c} + \mathbf{H}^{-1}\mathbf{u}_t$$

$$\Rightarrow \mathbf{y}_t = \boldsymbol{\mu} + [\boldsymbol{\Phi}(L)]^{-1}\mathbf{H}^{-1}\mathbf{u}_t$$

$$\mathbf{y}_t = \boldsymbol{\mu} + \mathbf{C}(L)\mathbf{u}_t$$

$$\Rightarrow \mathbf{C}(1) = [\boldsymbol{\Phi}(1)]^{-1}\mathbf{H}^{-1}$$

$$\mathbf{C}(1) = [\boldsymbol{\Phi}(1)]^{-1}\mathbf{H}^{-1}$$

$$\mathbf{C}(1)[\mathbf{C}(1)]' =$$

$$[\boldsymbol{\Phi}(1)]^{-1}\mathbf{H}^{-1}(\mathbf{H}^{-1})'\left\{[\boldsymbol{\Phi}(1)]^{-1}\right\}'$$

$\mathbf{C}(1)[\mathbf{C}(1)]' =$
$[\boldsymbol{\Phi}(1)]^{-1}\boldsymbol{\Omega}\{[\boldsymbol{\Phi}(1)]^{-1}\}'$

Can estimate: $\boldsymbol{\Phi}(1)$ and $\boldsymbol{\Omega}$ from VAR

Want: Lower triangular matrix $\mathbf{C}(1)$ such that

$$\mathbf{C}(1)[\mathbf{C}(1)]' = [\boldsymbol{\Phi}(1)]^{-1}\boldsymbol{\Omega}\{[\boldsymbol{\Phi}(1)]^{-1}\}'$$

Conclusion: $\mathbf{C}(1)$ is Cholesky factor of

$$[\boldsymbol{\Phi}(1)]^{-1}\boldsymbol{\Omega}\{[\boldsymbol{\Phi}(1)]^{-1}\}'$$

To get $\mathbf{H}$ we then use fact that

$$\mathbf{C}(1) = [\boldsymbol{\Phi}(1)]^{-1}\mathbf{H}^{-1}$$
$$\mathbf{H} = [\mathbf{C}(1)]^{-1}[\boldsymbol{\Phi}(1)]^{-1}$$

Summary:

(1) Estimate VAR's by OLS

$$\mathbf{y}_t = \begin{bmatrix} \Delta x_t \\ \Delta n_t \end{bmatrix}$$
$$\mathbf{y}_t = \mathbf{c} + \hat{\boldsymbol{\Phi}}_1\mathbf{y}_{t-1} + \hat{\boldsymbol{\Phi}}_2\mathbf{y}_{t-2} + \cdots + \hat{\boldsymbol{\Phi}}_p\mathbf{y}_{t-p} + \hat{\boldsymbol{\varepsilon}}_t$$
$$\hat{\boldsymbol{\Omega}} = T^{-1}\sum_{t=1}^{T}\hat{\boldsymbol{\varepsilon}}_t\hat{\boldsymbol{\varepsilon}}_t'$$

(2) Find Cholesky factor or lower triangular matrix $\widehat{\mathbf{C}}$ such that

$$\widehat{\mathbf{C}}\widehat{\mathbf{C}}' = \hat{\mathbf{Q}}\hat{\boldsymbol{\Omega}}\hat{\mathbf{Q}}'$$
$$\hat{\mathbf{Q}} = (\mathbf{I}_2 - \hat{\boldsymbol{\Phi}}_1 - \hat{\boldsymbol{\Phi}}_2 - \cdots - \hat{\boldsymbol{\Phi}}_p)^{-1}$$

(3) Technology shock and demand shock for date t are first and second elements of

$$\hat{\mathbf{u}}_t = \hat{\mathbf{B}}_0 \hat{\boldsymbol{\varepsilon}}_t$$

where

$$\hat{\mathbf{B}}_0 = \hat{\mathbf{C}}^{-1} \hat{\mathbf{Q}}$$

(4) Effect of tech shock or demand shock at date t on $\mathbf{y}_{t+s}$ are given by first and second columns, respectively, of

$$\frac{\partial \mathbf{y}_{t+s}}{\partial \mathbf{u}_t'} = \boldsymbol{\Psi}_s \mathbf{B}_0^{-1}$$

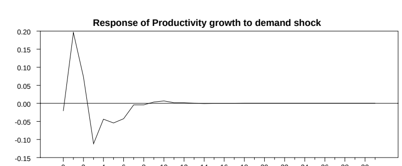
- system(model=basemod)
- variables delta_x delta_n
- lags 1 to 4
- det constant
- end(system)
- estimate

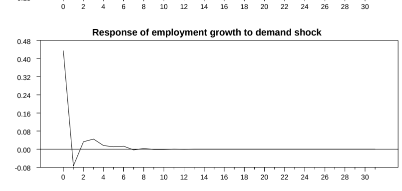
- display 'I - sum of VAR coeffs' %varlagsums
- compute Q = inv(%varlagsums)
- compute C = %decomp(Q*%sigma *tr(Q))
- compute Binv = %varlagsums*C

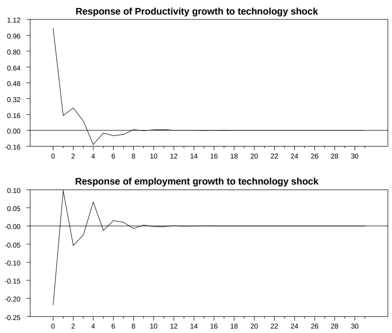
$$\hat{\boldsymbol{\Omega}} = T^{-1}\sum_{t=1}^{T} \hat{\boldsymbol{\varepsilon}}_t \hat{\boldsymbol{\varepsilon}}_t'$$
$$\hat{\mathbf{C}}\hat{\mathbf{C}}' = \hat{\mathbf{Q}}\hat{\boldsymbol{\Omega}}\hat{\mathbf{Q}}'$$
$$\hat{\mathbf{Q}} = (\mathbf{I}_2 - \hat{\boldsymbol{\Phi}}_1 - \hat{\boldsymbol{\Phi}}_2 - \cdots - \hat{\boldsymbol{\Phi}}_p)^{-1}$$
$$\hat{\mathbf{B}}_0 = \hat{\mathbf{C}}^{-1}\hat{\mathbf{Q}}$$
$$\hat{\mathbf{B}}_0^{-1} = (\mathbf{I}_2 - \hat{\boldsymbol{\Phi}}_1 - \hat{\boldsymbol{\Phi}}_2 - \cdots - \hat{\boldsymbol{\Phi}}_p)\mathbf{C}$$
$$\partial \mathbf{y}_{t+s}/\partial \mathbf{u}_t' = \boldsymbol{\Psi}_s \mathbf{B}_0^{-1}$$

- declare vect[labels] implabel(neqn)
- compute inlabel=|| 'technology shock', 'demand shock' ||
- compute outlabel=|| 'Productivity growth', 'employment growth' ||
- declare rect[series] impblk(neqn,1)

- smpl 1 nsteps
- do j=1,neqn
- impulse(model=basemod, shock=%xcol(binv,j), result=impblk) neqn nsteps
- spgraph(vfields=neqn)
- do i=1,neqn
- compute graphlabel='Response of ' +outlabel(1,i) +' to ' +inlabel(1,j)
- graph(header=graphlabel,nodates,number=0) 1
- # impblk(i,1)
- end do i
- spgraph(done)
- end do j







Drawbacks:

(1) $\hat{\mathbf{Q}} = (\mathbf{I}_2 - \hat{\boldsymbol{\Phi}}_1 - \hat{\boldsymbol{\Phi}}_2 - \cdots - \hat{\boldsymbol{\Phi}}_p)^{-1}$ is estimated poorly, sensitive to p

(2) technology shock could be temporary (e.g., delay in adoption of discovered technology)

(3) demand shock could be permanent (e.g., lost human capital)