Answer key for the 2004 final exam (econ 220b).

1. See the textbook.

2. This test is discussed on pages 121 and 122 of Hayashi. We assume that g has continuous first derivatives and then apply the delta method, exploiting the fact that g is approximately linear in a small neighborhood around the true parameter β. If $G(\beta)$ denotes the vector $\frac{\partial g}{\partial \beta}$ then the test statistic we should use is

$$g(\hat{\beta})' \left[G(\hat{\beta})' \hat{\sigma}^2 (X'X)^{-1} G(\hat{\beta}) \right]^{-1} g(\hat{\beta})$$

and this statistic converges in distribution under the null to a chi-square with one degree of freedom.

3. (a) No.

   (b) Yes.

   (c) This is fine. The errors form a homoskedastic martingale difference sequence, so the standard error in the t-test will converge to the true standard error. Moreover, the OLS coefficient will be asymptotically normal.

   (d) This will be fine too. The Newey-West standard error will also converge to the correct value.

   (e) In this setting the test based on the student t critical values should be slightly more accurate in most samples since its standard error is consistent and requires the estimation of fewer terms.

4. (a) For any sequence of random variables $\{x_t\}_{t=0}^{\infty}$, the value $\text{plim}_{t \to \infty} x_t$ is the χ constant (if it exists) such that, for any positive real number ϵ,

$$\lim_{t \to \infty} \text{Prob}[|x_t - \chi| > \epsilon] = 0.$$

This value may not exist.

These three expressions indicate that, in larger sample sizes, the sample average of z_t approaches zero, the second moment approaches the positive constant q, and the cross moment with u_t approaches the positive constant γ. For reasonable conditions on the moments of z and u, these limits are the population moments.

(b)

$$\begin{aligned}
\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix} &= \begin{pmatrix}1 & \bar{z}\\ \bar{z} & \bar{z^2}\end{pmatrix}^{-1}\begin{pmatrix}\bar{y}\\ \bar{zy}\end{pmatrix}\\
&\xrightarrow{p} \begin{pmatrix}1 & 0\\ 0 & 1/q\end{pmatrix}\begin{pmatrix}\alpha\\ \text{plim}\{\bar{z}+\beta\bar{z^2}+\bar{zu}\}\end{pmatrix}\\
&= \begin{pmatrix}\alpha\\ \beta\end{pmatrix} + \begin{pmatrix}0\\ \gamma/q\end{pmatrix}.
\end{aligned}$$

The usual OLS estimator will be an inconsistent estimator of β.

(c) We require the following:

$$\text{plim}\, T^{-1}\sum x_t z_t \neq 0 \quad \text{plim}\, T^{-1}\sum x_t x_t' = Q \quad \text{plim}\, T^{-1}\sum x_t u_t = 0$$

where Q has full rank.

(d) Let Z be the full matrix of regressors. The two-stage least squares estimate is

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix} = \left[Z'X(X'X)^{-1}X'Z\right]^{-1}Z'X(X'X)^{-1}X'Y$$

(e) The orthogonality condition is $E(x_t u_t) = 0$. Substituting for u_t gives

$$h = x_t(y_t - \alpha - \beta z_t).$$

5. (a) $\mathcal{L}(\theta; Y) = -(T/2)\log(2\pi/\sigma^2) - (1/2\sigma^2)\sum(y_t - \mu)^2$.

(b) The first-order conditions of this maximization require

$$\frac{\partial \mathcal{L}}{\partial \mu} = (1/\sigma^2)\sum(y_t - \mu) = 0 \qquad \Rightarrow \qquad \hat{\mu} = \bar{y}.$$
$$\frac{\partial \mathcal{L}}{\partial \sigma^2} = -T/(2\sigma^2) + (2/\sigma^4)\sum(y_t - \mu)^2 = 0 \quad \Rightarrow \quad \hat{\sigma}^2 = T^{-1}\sum(y_t - \hat{\mu})^2$$

(c) After carefully inspecting our work in part (b), it is clear that

$$h(\theta, w_t) = \begin{pmatrix} (y_t - \mu)/\sigma^2 \\ -1/2\sigma^2 + (y_t - \mu)^2/2\sigma^4 \end{pmatrix}$$

(d) Taking expectations gives the result immediately.

(e) First calculate $Eh(\theta, w_t)h(\theta, w_t)'$.

$$\begin{aligned} Eh(\theta, w_t)h(\theta, w_t)' &= \begin{pmatrix} E(y_t - \mu)^2/\sigma^4 & -E\big[(y_t - \mu)/2\sigma^3 + (y_t - \mu)^3/2\sigma^3\big] \\ \cdot & E\big[1/4\sigma^4 - (y_t - \mu)^2/2\sigma^6 + (y_t - \mu)^4/4\sigma^8\big] \end{pmatrix} \\ &= \begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 1/4\sigma^4 - 2/4\sigma^4 + 3/4\sigma^4 \end{pmatrix} \\ &= \begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 1/2\sigma^4 \end{pmatrix} \end{aligned}$$

(f) Taking derivatives gives

$$\begin{aligned} \hat{D}' &= T^{-1}\sum_{t=1}^{T} \begin{pmatrix} -1/\hat{\sigma}^2 & -(y_t - \hat{\mu})/\hat{\sigma}^4 \\ (y_t - \hat{\mu})/\hat{\sigma}^4 & 1/2\hat{\sigma}^4 - (y_t - \hat{\mu})^2/\hat{\sigma}^6 \end{pmatrix} \\ &= \begin{pmatrix} -1/\hat{\sigma}^2 & 0 \\ 0 & -1/2\hat{\sigma}^4 \end{pmatrix}. \end{aligned}$$

(g) The upper left element of $\hat{V}$ is the asymptotic variance of $\hat{\mu}$ and equals $1/\hat{\sigma}^2$ (use the plug-in estimator of S).

(h) The moment conditions we used for our GMM estimator are motivated by the Gaussian likelihood function, but can hold for other distributions too. For any such distribution, we can use the MLE for a normal distribution to find a consistent and asymptotically normal estimator of the mean. Other, more efficient, estimators might also exist.