

# Risk Premia in Crude Oil Futures Prices\*

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September 14, 2011

Revised: September 20, 2011

## Abstract

If commercial producers or financial investors use futures contracts to hedge against commodity price risk, the arbitrageurs who take the other side of the contracts may receive compensation for their assumption of nondiversifiable risk in the form of positive expected returns from their positions. We show that this interaction can produce an affine factor structure to commodity futures prices, and develop new algorithms for estimation of such models using unbalanced data sets in which the duration of observed contracts changes with each observation. We document significant changes in oil futures risk premia since 2005, with the compensation to the long position smaller on average but more volatile in more recent data. This observation is consistent with the claim that index-fund investing has become more important relative to commercial hedging in determining the structure of crude oil futures risk premia over time.

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\*Financial Support from the University of Chicago Booth School of Business is gratefully acknowledged.

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# 1 Introduction

Volatile oil prices have been drawing a lot of attention in recent years, with Hamilton (2009) for example suggesting they were a contributing factor in the recession of 2007-2009. There has also been considerable interest in whether there is any connection between this volatility and increased participation in commodity futures markets by financial investors. Recent investigations of the latter question include Tang and Xiong (2011), Buyuksahin and Robe (2011), Alquist and Gervais (2011), and Irwin and Sanders (2011).

One can characterize the observable characteristics of risk premia in commodity futures prices by looking empirically for predictability in the change in the price of a given futures contract. Alquist and Kilian (2010) summarized some of the recent evidence for such predictability, and Singleton (2011) and Mou (2010) showed that measures related to financial participation in futures markets may help statistically to explain these risk premia.

An old theme in the literature on commodity prices is Keynes (1930) theory of normal backwardation, according to which producers of the physical commodity want to hedge their price risk by selling futures contracts, and are willing to pay a premium for the privilege. If taking the other side of the contract exposes arbitrageurs to nondiversifiable risk, arbitrageurs will be compensated in equilibrium for insuring the commercial hedgers in the form of a futures price below the expected future spot price. Empirical support for this view has come from Carter, Rausser, and Schmitz (1983), Chang (1985), Bessembinder (1992), De Roon, Nijman, and Veld (2000), and Acharya, Lochstoer, and Ramadorai (2010), among others. Buying pressure from investors seeking portfolio diversification could exert a similar effect in the opposite direction, shifting the receipt of the risk premium from the long side to the short

side of the contract.

We explore this possibility using a framework in which dynamic variation in risk premia is modeled explicitly and priced consistently with no-arbitrage conditions across different contracts. We adapt the class of Gaussian affine term structure models originally developed by Vasicek (1977), Duffie and Kan (1996), Dai and Singleton (2002), Duffee (2002), and Ang and Piazzesi (2003) to characterize the relation between yields on bonds of different maturities. Related models have also been used to describe commodity futures prices by Schwartz (1997), Schwartz and Smith (2000), and Casassus and Collin-Dufresne (2006), among others.

This paper offers a number of methodological advances for use of these models to study commodity futures prices. First, we develop the basic relations directly for discrete-time observations, extending the contributions of Ang and Piazzesi (2003) to the setting of commodity futures prices. This allows a much more transparent mapping between model parameters and properties of observable OLS regressions. Second, we show how such relations could emerge from an explicit optimizing framework in which arbitrageurs interact with commercial producers seeking hedges or with financial investors seeking diversification. We do so by extending the models in Vayanos and Vila (2009) and Hamilton and Wu (forthcoming), which were originally used to describe how bond supplies affect relative yields, but are adapted in the current context to summarize how hedging demand would influence commodity futures prices. Third, we show how parameter estimates can be obtained directly from unbalanced data in which the remaining duration of observed contracts changes with each new observation, developing an alternative to the Kalman filter methodology used for this purpose by Cortazar and Naranjo (2006). Fourth, we show how the estimation method of Hamilton and Wu (2010)

provides diagnostic tools to reveal exactly where the model succeeds and where it fails to match the observed data.

We apply these methods to prices of crude oil futures contracts over 1990-2011. We document significant changes in risk premia in 2005 as the volume of futures trading began to grow significantly. While traders taking the long position in near contracts earned a positive return on average prior to 2005, that premium decreased substantially after 2005, becoming negative when the slope of the futures curve was high. This observation is consistent with the claim that historically commercial producers paid a premium to arbitrageurs for the privilege of hedging price risk, but in more recent periods financial investors have become natural counterparties for commercial hedgers. We also uncover seasonal variation of risk premia over the month, with changes as the nearest contract approaches expiry that cannot be explained from a shortening duration alone.

The plan of the paper is as follows. Section 2 develops the model, and Section 3 describes our approach to empirical estimation of parameters. Section 4 presents results for our baseline specification, while Section 5 presents results for a model allowing for more general variation as contracts near expiration. Conclusions are offered in Section 6.

## **2 Model.**

### **2.1 Role of arbitrageurs.**

Let  $F_{nt}$  denote the price of oil associated with an  $n$ -period futures contract entered into at date  $t$ . Following (Duffie, 1992, p. 39), we interpret a long position entered into at date  $t$

and closed at date  $t + 1$  as associated with a cash flow of zero at date  $t$  and  $F_{n-1,t+1} - F_{nt}$  at date  $t + 1$ . Let  $z_{nt}$  denote an arbitrageur's notional exposure (with  $z_{nt} > 0$  denoting a long position and  $z_{nt} < 0$  for short), so that  $z_{nt}/F_{nt}$  is the number of barrels purchased with  $n$ -period contracts. The arbitrageur's cash flow for period  $t + 1$  is then

$$W_{t+1} = \sum_{n=1}^N z_{nt} \frac{F_{n-1,t+1} - F_{nt}}{F_{nt}}.$$

Suppose that the arbitrageur chooses positions  $\{z_{1t}, \dots, z_{nt}\}$  so as to maximize

$$E_t(W_{t+1}) - (\gamma/2)\text{Var}_t(W_{t+1}). \quad (1)$$

We posit that the log of the price associated with any futures contract is an affine function of some vector of factors  $x_t$  which follow a Gaussian VAR:

$$x_{t+1} = c + \rho x_t + \Sigma u_{t+1} \quad u_t \sim \text{i.i.d. } N(0, I_m) \quad (2)$$

$$f_{nt} = \log F_{nt} = \alpha_n + \beta'_n x_t. \quad (3)$$

Using a similar approximation to that in Hamilton and Wu (forthcominga), we show in Appendix A that under this assumption,

$$E_t(W_{t+1}) \approx \sum_{n=1}^N z_{nt} [\alpha_{n-1} + \beta'_{n-1}(c + \rho x_t) - \alpha_n - \beta'_n x_t + (1/2)\beta'_{n-1}\Sigma\Sigma'\beta_{n-1}] \quad (4)$$

$$\text{Var}_t(W_{t+1}) \approx \left( \sum_{n=1}^N z_{nt} \beta'_{n-1} \right) \Sigma \Sigma' \left( \sum_{\ell=1}^N z_{\ell t} \beta_{\ell-1} \right). \quad (5)$$

The first-order conditions for the arbitrageur's positions satisfy

$$\frac{\partial E_t(W_{t+1})}{\partial z_{nt}} = (\gamma/2) \frac{\partial \text{Var}_t(W_{t+1})}{\partial z_{nt}},$$

which under (4) and (5) becomes

$$\alpha_{n-1} + \beta'_{n-1}(c + \rho x_t) - \alpha_n - \beta'_n x_t + (1/2) \beta'_{n-1} \Sigma \Sigma' \beta_{n-1} = \beta'_{n-1} \lambda_t \quad (6)$$

for

$$\lambda_t = \gamma \Sigma \Sigma' \left( \sum_{\ell=1}^N z_{\ell t} \beta_{\ell-1} \right). \quad (7)$$

Suppose we conjecture that in equilibrium the exposures  $z_{nt}$  selected by arbitrageurs are themselves affine functions of the vector of factors, so that

$$\lambda_t = \lambda + \Lambda x_t. \quad (8)$$

Then (6) requires

$$\beta'_n = \beta'_{n-1} \rho - \beta'_{n-1} \Lambda \quad (9)$$

$$\alpha_n = \alpha_{n-1} + \beta'_{n-1} c + (1/2) \beta'_{n-1} \Sigma \Sigma' \beta_{n-1} - \beta'_{n-1} \lambda. \quad (10)$$

Note that in the special case of risk-neutral arbitrageurs ( $\gamma = 0$ ), we would have  $\lambda = 0$  and  $\Lambda = 0$ .

The recursions (9) and (10) can equivalently be viewed as the equilibrium conditions that would result if risk-neutral arbitrageurs were to regard the factor dynamics as being governed not by (2) but instead by<sup>1</sup>

$$x_{t+1} = c^Q + \rho^Q x_t + \Sigma u_{t+1}^Q \quad (11)$$

$$c^Q = c - \lambda \quad (12)$$

$$\rho^Q = \rho - \Lambda \quad (13)$$

$$u_{t+1}^Q \stackrel{Q}{\sim} N(0, I_m).$$

## 2.2 Role of index-fund traders.

We illustrate how index-fund investment can influence futures prices in this setting. Suppose there are some investors who always want to have a long position in the 2-period contract, regardless of anything happening to fundamentals. At the start of each new period, these investors close out their previous position (which is now a 1-period contract) and replace it with a new long position in what is now the current 2-period contract. Let the scalar  $K_t$  denote the notional value of 2-period contracts that investors want to buy in period  $t$ , and suppose this evolves exogenously according to

$$K_t = c^K + \rho^K K_{t-1} + \Sigma^K u_t^K. \quad (14)$$

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<sup>1</sup>Our recursions (9) and (10) are essentially the same as equations (17) in Ang and Piazzesi (2003), with the important difference being that their recursion for the intercept adds a term  $\delta_0$  for each  $n$ , corresponding to the interest earned each period. No such term appears in our expression because there is no initial capital invested. Another minor notational difference is that our  $\lambda$  corresponds to their  $\Sigma\lambda_0$  while our  $\Lambda$  corresponds to their  $\Sigma\lambda_1$ . An advantage of our notation in the current setting is that our  $\lambda_t$  is then measured in the same units as  $x_t$  and is immediately interpreted as the direct adjustment to  $c$  and  $\rho$  that results from risk aversion by arbitrageurs.

If investors and arbitrageurs are the only participants in the market, then equilibrium futures prices must be such as to persuade arbitrageurs to take the opposite side of the investors. Thus arbitrageurs are always short the two-period contract, close that position when it becomes a 1-period contract, take the short side of the new 2-period contract, and have zero net exposure to any other contract in equilibrium. In other words, the process for  $\{f_{nt}\}_{n=0}^N$  must be such that (6) and (7) are satisfied with

$$z_{nt} = \begin{cases} -K_t & \text{for } n = 2 \\ 0 & \text{otherwise} \end{cases}.$$

From (7), this requires

$$\lambda_t = -\gamma \Sigma \Sigma' \beta_1 K_t. \quad (15)$$

Suppose that the spot price depends solely on a scalar “fundamentals” factor  $x_t^*$ :

$$f_{0t} = x_t^* \quad (16)$$

$$x_t^* = c^* + \rho^* x_{t-1}^* + \Sigma^* u_t^*.$$

We conjecture that in equilibrium, the factor  $x_t$  governing futures prices includes both fundamentals and the level of index-fund investment,  $x_t = (x_t^*, K_t)'$ , with (16) implying  $\beta_0 = (1, 0)$  and the factor evolving according to

$$x_t = c + \rho x_{t-1} + \Sigma u_t$$

or written out explicitly,

$$\begin{bmatrix} x_t^* \\ K_t \end{bmatrix} = \begin{bmatrix} c^* \\ c^K \end{bmatrix} + \begin{bmatrix} \rho^* & 0 \\ 0 & \rho^K \end{bmatrix} \begin{bmatrix} x_{t-1}^* \\ K_{t-1} \end{bmatrix} + \begin{bmatrix} \Sigma^* & 0 \\ 0 & \Sigma^K \end{bmatrix} \begin{bmatrix} u_t^* \\ u_t^K \end{bmatrix}.$$

We can then recognize (15) as a special case of (8) with

$$\lambda = 0$$

$$\Lambda_{(2 \times 2)} = \begin{bmatrix} 0 & -\gamma \Sigma \Sigma' \beta_1 \end{bmatrix}.$$

Hence

$$\rho^Q = \begin{bmatrix} \rho^* & 0 \\ 0 & \rho^K \end{bmatrix} + \begin{bmatrix} 0 & \gamma \Sigma \Sigma' \beta_1 \end{bmatrix} \quad (17)$$

$$\begin{aligned} \beta_1' &= \beta_0' \rho^Q \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \rho^Q \\ &= \begin{bmatrix} \rho^* & \gamma \left[ (\Sigma^*)^2 & 0 \right] \beta_1 \end{bmatrix} \end{aligned}$$

$$\beta_1 = \begin{bmatrix} \rho^* \\ \gamma \rho^* (\Sigma^*)^2 \end{bmatrix}. \quad (18)$$

Assuming  $\rho^* > 0$  and  $K_t > 0$ , the effect of index-fund buying of the 2-period contract is also to increase the price of a 1-period contract. The reason is that the 2-period contract that the arbitrageurs are currently being induced to short exposes the arbitrageurs to risk associated

with uncertainty about the value of  $x_{t+1}^*$ . The 1-period contract is also exposed to risk from  $x_{t+1}^*$ . If a 1-period contract purchased at  $t$  provided zero expected return, arbitrageurs would want to go long the 1-period contract in order to diversify their risk associated with being short the 2-period contract. But there is no counterparty who wants to short the 1-period contract, so equilibrium requires a price  $f_{1t}$  such that someone shorting the 1-period contract would also have a positive expected return, earned in the form of a higher price for  $f_{1t}$ .

Substituting (18) into (17), we now know  $\rho^Q$  and can calculate  $\beta_n = (\rho^Q)^n \beta_0$  for each  $n$ . Thus investment buying does not matter for  $f_{0t}$  but does affect every  $f_{nt}$  for  $n > 0$ , through the same mechanism as operates on the 1-period contract. In particular, from (6),

$$E_t \left( \frac{F_{n-1,t+1} - F_{nt}}{F_{nt}} \right) \approx \beta'_{n-1} \lambda_t = -\gamma \beta'_1 (\rho^Q)^{n-2} \Sigma \Sigma' \beta_1 K_t,$$

which in general has the opposite sign of  $K_t$  for all  $n$ ; someone would earn a positive expected return by taking the short position in a contract of any duration.

### 2.3 Empirical implementation.

For purposes of empirical estimation we interpret  $t$  as describing weekly intervals. This allows us to capture some key calendar regularities in the data without introducing an excessive number of parameters. NYMEX crude oil futures contracts expire on the third business day prior to the 25th calendar day of the month prior to the month on which the contract is written. To preserve the important calendar structure of the raw data, we divide the “month” leading up to a contract expiry into four “weeks”, defined as follows:

week 1 ends on the last business day of the previous calendar month

week 2 ends on the 5th business day of the current calendar month

week 3 ends on the 10th business day of the current calendar month

week 4 ends on the day when the near contract expires

Associated with any week  $t$  is an indicator  $j_t \in \{1, 2, 3, 4\}$  of where in the month week  $t$  falls.

Our estimation uses the nearest three contracts. If we interpret the price at expiry as an  $n = 0$  week-ahead contract, the observation  $y_t$  for week  $t$  would be characterized using the notation of Section 2 as follows:

$$y_t = \begin{cases} (f_{3t}, f_{7t}, f_{11,t})' & \text{if } j_t = 1 \\ (f_{2t}, f_{6t}, f_{10,t})' & \text{if } j_t = 2 \\ (f_{1t}, f_{5t}, f_{9t})' & \text{if } j_t = 3 \\ (f_{0t}, f_{4t}, f_{8t})' & \text{if } j_t = 4 \end{cases} .$$

Table 1 summarizes the relation between the weekly indicator ( $j$ ), months until expiry is reached ( $k$ ), and weeks remaining until expiry ( $n$ ). This feature that the maturity of observed contracts changes with each observation  $t$  is one reason that much of the research with commodity futures contracts has used monthly data. However, in our application a key interest is in the higher-frequency movements and specific calendar effects. Fortunately, the framework developed in Section 2 gives us an exact description of the likelihood function for the data as

actually observed, as we now describe.

We will assume that there are two underlying factors (that is,  $x_t$  is  $2 \times 1$ ). Since (3) implies that each element of the  $(3 \times 1)$  vector  $y_t$  could be written as an exact linear function of  $x_t$ , the system as written is stochastically singular— according to the model, the third element of  $y_t$  should be given by an exact linear combination of the first two. This issue also commonly arises in studies of the term structure of interest rates. A standard approach in that literature<sup>2</sup> is to assume that some elements or linear combinations of  $y_t$  differ from the magnitude predicted in (3) by a measurement or specification error. In the results reported below, we assume that the  $k = 1$ - and  $2$ - month contracts are priced exactly as the model predicts. It is helpful for purposes of interpreting parameter estimates to summarize the information in these contracts in terms of the average level of the two prices, which we will associate with the first factor in the system, and spread between them, which we will associate with the second factor:

$$y_{1t} = H_1 y_t$$

$$H_1 = \begin{bmatrix} 0 & (1/2) & (1/2) \\ 0 & -1 & 1 \end{bmatrix}. \quad (19)$$

The two elements of  $y_{1t}$  are plotted in Figure 1.

We assume that the model correctly characterizes these two observed magnitudes. Since

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<sup>2</sup>See for example Chen and Scott (1993), Ang and Piazzesi (2003) and Joslin, Singleton, and Zhu (forthcoming). The observable implications of this assumption are explored in detail in Hamilton and Wu (forthcomingb).

$y_t = (f_{4-j_t}, f_{8-j_t}, f_{12-j_t})'$ , this implies that

$$y_{1t} = A_{1,j_t} + B_{1,j_t}x_t \quad (20)$$

$$A_{1,j_t} = H_1 \begin{bmatrix} \alpha_{4-j_t} \\ \alpha_{8-j_t} \\ \alpha_{12-j_t} \end{bmatrix} \quad \text{for } j_t = 1, 2, 3, \text{ or } 4$$

$$B_{1,j_t} = H_1 \begin{bmatrix} \beta'_{4-j_t} \\ \beta'_{8-j_t} \\ \beta'_{12-j_t} \end{bmatrix} \quad \text{for } j_t = 1, 2, 3, \text{ or } 4. \quad (21)$$

We will use the notational convention that if  $j_t = 1$ , then  $A_{1,j_{t-1}} = A_{14}$ .

If  $B_{1j}$  is invertible, the dynamics of the observed vector  $y_{1t}$  can be characterized by substituting (20) into (2):

$$y_{1t} = A_{1,j_t} + B_{1,j_t}c + B_{1,j_t}\rho \left[ B_{1,j_{t-1}}^{-1} (y_{1,t-1} - A_{1,j_{t-1}}) \right] + B_{1,j_t}\Sigma u_t.$$

Since  $u_t$  is independent of  $\{y_{t-1}, y_{t-2}, \dots, y_0\}$ , this means that the density of  $y_{1t}$  conditional on all previous observations is characterized by a VAR(1) with seasonally varying parameters:

$$y_{1t}|y_{t-1}, y_{t-2}, \dots, y_0 \sim N(\phi_{j_t} + \Phi_{j_t}y_{1,t-1}, \Omega_{j_t}) \quad (22)$$

$$\Omega_{j_t} = B_{1,j_t}\Sigma\Sigma'B'_{1,j_t}$$

$$\Phi_{j_t} = B_{1,j_t} \rho B_{1,j_t-1}^{-1}$$

$$\phi_{j_t} = A_{1,j_t} + B_{1,j_t} c - \Phi_{j_t} A_{1,j_t-1}.$$

Note that the predicted seasonal parameter variation arises from the fact that the number of weeks remaining until expiry of the observed contracts changes with each new week.

We postulate that the nearest contract, which we write as

$$y_{2t} = H_2 y_t$$

$$H_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad (23)$$

differs from the value predicted by the framework by a measurement or specification error with mean zero and variance  $\sigma_{e,j_t}^2$ :

$$y_{2t} = A_{2,j_t} + B_{2,j_t} x_t + \sigma_{e,j_t} u_{e,t}$$

$$A_{2,j_t} = H_2 \begin{bmatrix} \alpha_{4-j_t} \\ \alpha_{8-j_t} \\ \alpha_{12-j_t} \end{bmatrix} \quad \text{for } j = 1, 2, 3, 4$$

(1×1)

$$B_{2,j_t} = H_2 \begin{bmatrix} \beta'_{4-j_t} \\ \beta'_{8-j_t} \\ \beta'_{12-j_t} \end{bmatrix} \quad \text{for } j_t = 1, 2, 3, \text{ or } 4.$$

(1×2)

If the measurement error  $u_{et}$  is independent of past observations, this gives the conditional

distribution

$$y_{2t} | y_{1t}, y_{t-1}, y_{t-2}, \dots, y_0 \sim N(\gamma_{jt} + \Gamma_{jt} y_{1t}, \sigma_{e,jt}^2) \quad (24)$$

$$\Gamma_{jt} = B_{2,jt} B_{1,jt}^{-1} \quad (25)$$

$$\gamma_{jt} = A_{2,jt} - \Gamma_{jt} A_{1,jt}.$$

The density of  $y_t$  conditional on its own past history is thus the product of (22) with (24), meaning that the log likelihood for the full sample of observations  $(y'_T, y'_{T-1}, \dots, y'_1)'$  conditional on the initial observation  $y_0$  is given by

$$\mathcal{L} = \sum_{t=1}^T [\log g(y_{1t}; \phi_{jt} + \Phi_{jt} y_{1,t-1}, \Omega_{jt}) + \log g(y_{2t}; \gamma_{jt} + \Gamma_{jt} y_{1t}, \sigma_{e,jt}^2)] \quad (26)$$

where  $g(y; \mu, \Omega)$  denotes the multivariate Normal density with mean  $\mu$  and variance  $\Omega$  evaluated at the point  $y$ .

### 3 Estimation.

#### 3.1 Unrestricted reduced form.

The traditional approach to estimation of these kind of models would be to maximize the likelihood function with respect to the unknown structural parameters. However, Hamilton and Wu (2010) demonstrate that there can be big benefits from using an estimator that turns out to be asymptotically equivalent to MLE but is derived from simple OLS regressions. To understand this estimator, consider first how we would maximize the likelihood if we thought

of  $\phi_j, \Phi_j, \Omega_j, \gamma_j, \Gamma_j$ , and  $\sigma_{e_j}$  in the above representation as completely unrestricted parameters rather than the particular values implied by the structural model presented above. From this perspective, the log likelihood (26) could be written

$$\begin{aligned} & \mathcal{L}(\phi_1, \Phi_1, \Omega_1, \gamma_1, \Gamma_1, \sigma_{e1}, \dots, \phi_4, \Phi_4, \Omega_4, \gamma_4, \Gamma_4, \sigma_{e4}) \\ &= \sum_{j=1}^4 \mathcal{L}_{1j}(\phi_j, \Phi_j, \Omega_j) + \sum_{j=1}^4 \mathcal{L}_{2j}(\gamma_j, \Gamma_j, \sigma_{e_j}) \end{aligned} \quad (27)$$

$$\mathcal{L}_{1j}(\phi_j, \Phi_j, \Omega_j) = \sum_{t=1}^T \delta(j_t = j) \log g(y_{1t}; \phi_j + \Phi_j y_{1,t-1}, \Omega_j)$$

$$\begin{aligned} \log g(y_{1t}; \phi_j + \Phi_j y_{1,t-1}, \Omega_j) &= -\log 2\pi - (1/2) \log |\Omega_j| \\ &\quad - (1/2) (y_{1t} - \phi_j - \Phi_j y_{1,t-1})' \Omega_j^{-1} (y_{1t} - \phi_j - \Phi_j y_{1,t-1}) \end{aligned}$$

$$\mathcal{L}_{2j}(\gamma_j, \Gamma_j, \sigma_{e_j}) = \sum_{t=1}^T \delta(j_t = j) \log g(y_{2t}; \gamma_j + \Gamma_j y_{1t}, \sigma_{e_j}^2)$$

$$\log g(y_{2t}; \gamma_j + \Gamma_j y_{1t}, \sigma_{e_j}^2) = -\log 2\pi - (1/2) \log \sigma_{e_j}^2 - \frac{(y_{2t} - \gamma_j - \Gamma_j y_{1t})^2}{2\sigma_{e_j}^2}$$

where for example  $\delta(j_t = 1)$  is 1 if  $t$  is in the first week of the month and is zero otherwise. It is clear that the unconstrained likelihood function is in fact maximized by a series of OLS regressions. To estimate the parameters in block  $j$ , we collect all observations whose left-hand variable is in the  $j$ th week of the month, and simply perform OLS regressions on what now looks like a monthly data set.

Specifically, to estimate  $(\phi_j, \Phi_j, \Omega_j)$  for a particular  $j$ , we associate month  $\tau$  with an observed monthly-frequency vector  $y_{1,j,\tau}^\dagger$  defined as follows. For illustration, consider  $j = 1$  and

suppose that  $\tau$  corresponds to the month spanned by the last week of December and first 3 weeks of January. The first element of  $y_{1,1,\tau|\text{month}(\tau)=\text{Jan}}^\dagger$  is the average of the log prices of the March and April contracts as of the last business day of December. The second element of  $y_{1,1,\tau|\text{month}(\tau)=\text{Jan}}^\dagger$  is based on the log price of the April contract on the last day of December minus the log price of March contract. For  $j = 1$  and general  $\tau$ ,

$$y_{1,1,\tau}^\dagger = H_1 \begin{bmatrix} f_{3,t(\tau)} \\ f_{7,t(\tau)} \\ f_{11,t(\tau)} \end{bmatrix}$$

for  $H_1$  given in (19) and where  $t(\tau)$  denotes the week  $t$  associated with month  $\tau$ . The explanatory variables in these  $j = 1$  block regressions consist of a constant, the average log prices of the February and March contracts on the day in December when the January contract expired, and the spread between the March price and February price at the December expiry of the January contract:

$$x_{1,1,\tau}^\dagger = \begin{bmatrix} 1 \\ H_1 \begin{bmatrix} f_{0,t(\tau)-1} \\ f_{4,t(\tau)-1} \\ f_{8,t(\tau)-1} \end{bmatrix} \end{bmatrix}. \quad (28)$$

Consider the estimates from OLS regression of  $y_{1,1,\tau}^\dagger$  on  $x_{1,1,\tau}^\dagger$ ,

$$\begin{bmatrix} \hat{\phi}_1 & \hat{\Phi}_1 \end{bmatrix} = \left( \sum_{\tau=1}^{\mathcal{T}} y_{1,1,\tau}^\dagger x_{1,1,\tau}^{\dagger\prime} \right) \left( \sum_{\tau=1}^{\mathcal{T}} x_{1,1,\tau}^\dagger x_{1,1,\tau}^{\dagger\prime} \right)^{-1}$$

$$\hat{\Omega}_1 = \mathcal{T}^{-1} \sum_{\tau=1}^{\mathcal{T}} \left( y_{1,1,\tau}^\dagger - \begin{bmatrix} \hat{\phi}_1 & \hat{\Phi}_1 \end{bmatrix} x_{1,1,\tau}^\dagger \right) \left( y_{1,1,\tau}^\dagger - \begin{bmatrix} \hat{\phi}_1 & \hat{\Phi}_1 \end{bmatrix} x_{1,1,\tau}^\dagger \right)'$$

where  $\mathcal{T}$  denotes the number of months in the sample. These estimates maximize the log likelihood (27) with respect to  $\{\phi_1, \Phi_1, \Omega_1\}$ .

For  $j = 2$  we regress  $y_{1,2,\tau}^\dagger$  (whose first element, for example, would be the average of the March and April contracts as of the fifth business day in January) on  $x_{1,2,\tau}^\dagger$  (e.g., a constant and the level and spread as of the last day of December),

$$\begin{bmatrix} \hat{\phi}_2 & \hat{\Phi}_2 \end{bmatrix} = \left( \sum_{\tau=1}^{\mathcal{T}} y_{1,2,\tau}^\dagger x_{1,2,\tau}^{\dagger'} \right) \left( \sum_{\tau=1}^{\mathcal{T}} x_{1,2,\tau}^\dagger x_{1,2,\tau}^{\dagger'} \right)^{-1}$$

$$\hat{\Omega}_2 = \mathcal{T}^{-1} \sum_{\tau=1}^{\mathcal{T}} \left( y_{1,2,\tau}^\dagger - \begin{bmatrix} \hat{\phi}_2 & \hat{\Phi}_2 \end{bmatrix} x_{1,2,\tau}^\dagger \right) \left( y_{1,2,\tau}^\dagger - \begin{bmatrix} \hat{\phi}_2 & \hat{\Phi}_2 \end{bmatrix} x_{1,2,\tau}^\dagger \right)'$$

to obtain  $\hat{\phi}_2$ ,  $\hat{\Phi}_2$ , and  $\hat{\Omega}_2$ . Similar separate monthly regressions of the 1- and 2-month prices in the third or fourth week of each month on their values the week before produce  $\{\hat{\phi}_j, \hat{\Phi}_j, \hat{\Omega}_j\}$  for  $j = 3$  or 4.

Likewise, note that the components of  $\sum_{j=1}^4 \mathcal{L}_{2j}(\gamma_j, \Gamma_j, \sigma_e)$  take the form of regressions in which the residuals are uncorrelated across blocks, meaning full-information maximum likelihood estimates of  $\gamma_j$  and  $\Gamma_j$  are obtained by OLS regressions for individual  $j$ . For example, for  $j = 1$  and  $\tau$  corresponding to December-January,  $y_{2,j,\tau}^\dagger$  is the price of the February contract on the last day of December,

$$y_{2,1,\tau}^\dagger = H_2 \begin{bmatrix} f_{3,t(\tau)} \\ f_{7,t(\tau)} \\ f_{11,t(\tau)} \end{bmatrix},$$

for  $H_2$  in (23) and explanatory variables the level and slope as of the last day of December:

$$x_{2,1,\tau}^\dagger = \begin{bmatrix} 1 \\ H_1 \begin{bmatrix} f_{3,t(\tau)} \\ f_{7,t(\tau)} \\ f_{11,t(\tau)} \end{bmatrix} \end{bmatrix}.$$

The maximum likelihood estimates are given by

$$\begin{aligned} \begin{bmatrix} \hat{\gamma}_j & \hat{\Gamma}_j \end{bmatrix} &= \left( \sum_{\tau=1}^T y_{2,j,\tau}^\dagger x_{2,j,\tau}^{\dagger'} \right) \left( \sum_{\tau=1}^T x_{2,j,\tau}^\dagger x_{2,j,\tau}^{\dagger'} \right)^{-1} \quad \text{for } j = 1, 2, 3, 4 \\ \hat{\sigma}_{ej}^2 &= T^{-1} \sum_{\tau=1}^T \left( y_{2,j,\tau}^\dagger - \begin{bmatrix} \hat{\gamma}_j & \hat{\Gamma}_j \end{bmatrix} x_{2,j,\tau}^\dagger \right)^2. \end{aligned} \quad (29)$$

### 3.2 Structural estimation of the baseline model.

Now consider estimation of the underlying structural parameters of the model presented in Section 2. The key point to note is that the above OLS estimates  $\{\hat{\phi}_1, \hat{\Phi}_1, \hat{\Omega}_1, \hat{\gamma}_1, \hat{\Gamma}_1, \hat{\sigma}_{e1}, \dots, \hat{\phi}_4, \hat{\Phi}_4, \hat{\Omega}_4, \hat{\gamma}_4, \hat{\Gamma}_4, \hat{\sigma}_{e4}\}$  are sufficient statistics for inference about these parameters— anything that the full sample of data is able to tell us about the model parameters can be summarized by the values of these OLS estimates. The idea behind the minimum-chi-square estimation proposed by Hamilton and Wu (2010) is to choose structural parameters that would imply reduced-form coefficients as close as possible to the unrestricted estimates, an approach that turns out to be asymptotically equivalent to full MLE.

Note that the model developed here specifies observed prices in terms of an unobserved

factor vector  $x_t$ . There is an arbitrary normalization in any such system, in that if we were to multiply  $x_t$  by a nonsingular matrix and add a constant, the result would be observationally equivalent in terms of the implied likelihood for observed  $y_t$ .<sup>3</sup> Since we have treated the factors  $x_t$  as directly inferable from the values of  $y_{1t}$ , we normalize the factors so that they could be interpreted as the level and slope as of the date of expiry of the near-term contract:

$$x_t = H_1 y_t \quad \text{for } j_t = 4. \quad (30)$$

Recalling (20), this would be the case if

$$H_1 y_t = H_1 \begin{bmatrix} \alpha_0 \\ \alpha_4 \\ \alpha_8 \end{bmatrix} + H_1 \begin{bmatrix} \beta'_0 \\ \beta'_4 \\ \beta'_8 \end{bmatrix} x_t \quad \text{for } j_t = 4. \quad (31)$$

Substituting (30) into (31), our chosen normalization thus calls for

$$x_t = H_1 \begin{bmatrix} \alpha_0 \\ \alpha_4 \\ \alpha_8 \end{bmatrix} + H_1 \begin{bmatrix} \beta'_0 \\ \beta'_4 \\ \beta'_8 \end{bmatrix} x_t \quad \text{for } j_t = 4$$

$$H_1 \begin{bmatrix} \beta'_0 \\ \beta'_4 \\ \beta'_8 \end{bmatrix} = I_2 \quad (32)$$

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<sup>3</sup>For further discussion of identification and normalization, see Hamilton and Wu (2010).

$$H_1 \begin{bmatrix} \alpha_0 \\ \alpha_4 \\ \alpha_8 \end{bmatrix} = 0. \quad (33)$$

Since

$$\begin{bmatrix} f_{0t} \\ f_{4t} \\ f_{8t} \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \alpha_4 \\ \alpha_8 \end{bmatrix} + \begin{bmatrix} \beta'_0 \\ \beta'_4 \\ \beta'_8 \end{bmatrix} x_t,$$

our normalization could alternatively be described as  $x_t = H_1(f_{0t}, f_{4t}, f_{8t})'$  for all  $t$ . Following Joslin, Singleton, and Zhu (forthcoming) and Hamilton and Wu (forthcomingb), this can be implemented by defining  $\xi_1$  and  $\xi_2$  to be the eigenvalues of  $\rho^Q = \rho - \Lambda$ . Given this normalization and values for  $\xi_1$ ,  $\xi_2$ ,  $\Sigma$ , and  $\alpha_0$ , we can then determine the values for  $\rho^Q$ ,  $c^Q$ , and  $\{\beta_n, \alpha_n\}_{n=0}^N$ ; details are provided in Appendix B. These along with  $\rho$ ,  $c$ , and  $\sigma_e$  then provide everything we need to evaluate the likelihood function or to calculate what the predicted values for any of the unrestricted reduced-form coefficients ought to be.

Let  $\theta$  denote the vector of unknown structural parameters, that is, the 16 elements of  $\{\xi_1, \xi_2, \Sigma, \alpha_0, \rho, c, \sigma_{e1}, \sigma_{e2}, \sigma_{e3}, \sigma_{e4}\}$  for  $\Sigma$  lower triangular. Collect elements of the unrestricted OLS estimates in a vector  $\hat{\pi}$ :

$$\begin{aligned} \hat{\pi} &= (\hat{\pi}'_{\Phi}, \hat{\pi}'_{\Omega}, \hat{\pi}'_{\Gamma}, \hat{\pi}'_{\sigma})' \\ \hat{\pi}_{\Phi} &= \left( \left[ \text{vec} \left( \begin{bmatrix} \hat{\phi}_1 & \hat{\Phi}_1 \end{bmatrix} \right)' \right]', \dots, \left[ \text{vec} \left( \begin{bmatrix} \hat{\phi}_4 & \hat{\Phi}_4 \end{bmatrix} \right)' \right]' \right)' \\ \hat{\pi}_{\Omega} &= \left( \left[ \text{vech} \left( \hat{\Omega}_1 \right)' \right]', \dots, \left[ \text{vech} \left( \hat{\Omega}_4 \right)' \right]' \right)' \end{aligned}$$

$$\hat{\pi}_\Gamma = \left( \left[ \text{vec} \left( \begin{bmatrix} \hat{\gamma}_1 & \hat{\Gamma}_1 \end{bmatrix}' \right) \right]', \dots, \left[ \text{vec} \left( \begin{bmatrix} \hat{\gamma}_4 & \hat{\Gamma}_4 \end{bmatrix}' \right) \right]' \right)'$$

$$\hat{\pi}_\sigma = (\hat{\sigma}_{e1}, \hat{\sigma}_{e2}, \hat{\sigma}_{e3}, \hat{\sigma}_{e4})'.$$

Let  $g(\theta)$  denote the corresponding predicted values for those coefficients from the model; specific values for the elements of  $g(\theta)$  are summarized in Appendix C. The minimum-chi-square (MCS) estimate of  $\theta$  is the value that minimizes

$$\mathcal{T}[\hat{\pi} - g(\theta)]' \hat{R}[\hat{\pi} - g(\theta)] \quad (34)$$

for  $\hat{R}$  the information matrix associated with the OLS estimates  $\hat{\pi}$ , which is also detailed in Appendix C. The MCS estimator has the same asymptotic distribution as the maximum likelihood estimator, but has a number of computational and interpretive advantages over MLE discussed in Hamilton and Wu (2010). Because  $\hat{R}$  is block-diagonal with respect to  $\pi_\sigma$ , the MCS estimates of these parameters are given immediately by the OLS estimates (29). Hamilton and Wu (2010) show that asymptotic standard errors can be estimated using

$$E(\hat{\theta} - \theta_0)(\hat{\theta} - \theta_0)' \simeq \mathcal{T}^{-1}(\hat{G}' \hat{R} \hat{G})^{-1}$$

$$\hat{G} = \left. \frac{\partial g(\theta)}{\partial \theta'} \right|_{\theta=\hat{\theta}},$$

which are identical to the usual asymptotic errors that would be obtained by taking second derivatives of the log likelihood function (26) with respect to  $\theta$ .

## 4 Empirical results for the baseline model.

Crude oil futures contracts were first traded on the New York Mercantile Exchange (NYMEX) in 1983. In the first few years, volume was much lighter than the more recent data, and we choose to begin our empirical analysis in January, 1990. Figure 2 plots the total open interest on all NYMEX light sweet crude contracts. Volume expanded very quickly after 2004, in part in response to the increased purchases of futures contracts as a vehicle for financial diversification. Some researchers have suggested that participation in the markets by this new class of traders resulted in significant changes in the dynamic behavior of crude oil futures prices.<sup>4</sup> A likelihood ratio test (e.g., (Hamilton, 1994, p. 296)) of the null hypothesis that the coefficients of the unrestricted reduced form are constant over time against the alternative that all 52 parameters changed in January 2005 produces a  $\chi^2(52)$  statistic of 181.96, which calls for dramatic rejection of the null hypothesis ( $p$  value of  $2.2 \times 10^{-16}$ ). Since one of our interests in this paper is to document how futures price dynamics have changed over time, we conduct our analysis on two subsamples, the first covering January 1990 through December 2004, and the second January 2005 through June 2011.

The first panel of Table 2 reports minimum-chi-square estimates of the 16 elements of  $\theta$  based on the first subsample. The eigenvalues of  $\rho$ , the matrix summarizing the objective  $P$ -measure persistence of factors, are 0.9956 and 0.9319, implying that both level and slope are highly persistent, with similar estimates for their  $Q$ -measure counterparts ( $\xi_1$  and  $\xi_2$ ).

The differences between the  $P$ - and  $Q$ -measures, or implied characterization of  $\lambda_t$ , are reported in the second panel of Table 2.<sup>5</sup> The individual elements of  $\lambda$  and  $\Lambda$  are generally

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<sup>4</sup>See for example Singleton (2011), Tang and Xiong (2011), Mou (2010), and Buyuksahin and Robe (2011).

<sup>5</sup>Standard errors for  $\lambda$  and  $\Lambda$  were obtained by reparameterizing the MCS estimation in terms of  $\lambda$  and  $\Lambda$

small and statistically insignificant. The last two entries of Table 2 report the elements of  $\lambda + \Lambda \bar{x}$ , where  $\bar{x}$  is the average value for the level and spread over the sample. The positive value of 0.0037 for the first element of this vector suggests that an investor who was always long in the two contracts would on average have come out ahead over this period, an estimate that is just statistically significant at the 5% level.

Table 3 reports parameter estimates for the later subsample, in which there appear to be significant differences in risk pricing from the earlier data. Most noteworthy is the large negative value for  $\Lambda_{12}$ . This signifies that when the spread (the second element of  $x_t$ ) gets sufficiently high, a long position in the 1- and 2-month contracts would on average lose money. We also see from the last entry of Table 3 that the first element of  $\lambda + \Lambda \bar{x}$  is smaller in the second subsample than in the first, and is no longer statistically significant. The average reward for taking long positions in the second subsample is not as evident in the first subsample.

Figure 3 plots our estimated values for  $\lambda_t = \lambda + \Lambda x_t$  for each week  $t$  in our sample, along with 95% confidence intervals. The price of level risk (top panel) was uniformly positive up until 2006, but has often been negative since 2008. By contrast, slope risk (bottom panel) was typically not priced before 2004, whereas going long the 2-month contract and short the 1-month has frequently been associated with positive expected returns since then.

Following Cochrane and Piazzesi (2009) and Bauer, Rudebusch, and Wu (2011), another way to summarize the implications of these results is to calculate how different the log price of a given contract would be if there was no compensation for risk. To get this number, we calculate  $\tilde{f}_{nt} = \tilde{\alpha}_n + \tilde{\beta}_n x_t$ , where  $\tilde{\beta}_n$  and  $\tilde{\alpha}_n$  denote the values that would be obtained from the 

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 instead of  $c$  and  $\rho$ . The values of  $\lambda$  and  $\Lambda$  can be obtained analytically from  $\lambda = c - c^Q$  and  $\Lambda = \rho - \rho^Q$  with  $c^Q$  given by equation(46) and  $\rho^Q$  given by equation (41).

recursions (9) and (10) if  $\Lambda$  and  $\lambda$  were both set to zero. The value for the difference  $\tilde{f}_{nt} - f_{nt}$  for  $n = 8$  weeks is plotted in Figure 4. In the absence of risk effects, an 8-week contract price would have been a few percent higher on average over the 1990-2004 subsample. Since 2005, risk aversion has made a more volatile contribution, though the average effect is significantly smaller.

In terms of the framework proposed in Section 2 for interpreting these results, the positive average value for the first element of  $\lambda_t$  in the first subsample suggests that arbitrageurs were on average long in crude oil futures contracts over this period, accepting the positive expected earnings from their positions as compensation for providing insurance to sellers, who were presumably commercial producers who wanted to hedge their price risks by selling futures contracts. From that perspective, an increase in index fund buying could have been one explanation for why a long position in futures contracts no longer has a statistically significant positive return. In effect, index-fund buyers are serving as counterparty for commercial hedgers, and are willing to do so without the risk compensation that the position earned on average in the first subsample. The emerging positive return to a spreading position (positive average second element of  $\lambda_t$  in the second subsample) would be consistent with the view that arbitrageurs are buying and holding 2-month futures from oil producers, but then selling these positions and going short 1-month futures as they sell to index-fund investors.

As noted by Hamilton and Wu (2010), another benefit of estimation by minimum chi square is that the optimized value for the objective function provides an immediate test of the overall framework. Under the null hypothesis that the model is correctly specified, the minimum value achieved for (34) has an asymptotic  $\chi^2$  distribution with degrees of freedom

given by the number of overidentifying restrictions. The first column of Table 4 reports the value of this statistic for each of the two subsamples. The model is overwhelmingly rejected in either subsample.

Because the weighting matrix  $\hat{R}$  in (48) is block-diagonal, it is easy to decompose these test statistics into components coming from the respective elements of  $\pi$ , as is done in subsequent columns of Table 4. In the first subsample, about half of the value of the test statistic comes from the  $\pi_\Omega$  block— the differences in the variability of the level and slope across different weeks of the month is more than can be explained by the fact that the maturities of observed contracts are changing week to week. The biggest problem in the second subsample come from the  $\pi_\Phi$  block— unrestricted forecasts of the level and slope vary more week-to-week than is readily explained by differences in the maturities of the contracts.

It is also possible to look one parameter at a time at where the structural model misses. For each of the unrestricted reduced-form parameters  $\pi$  there is a corresponding prediction from the model  $g(\theta)$  for what that value is supposed to be if the model is correct. Figures 5 and 6 plot the unrestricted OLS estimates of the various elements of  $\pi$  along with their 95% confidence intervals for the first subsample. The thick red lines indicate the value the coefficient is predicted to have according to the structural parameters reported in Table 2. The biggest problems come from the fact that the model underpredicts the difficulty of forecasting the spread in weeks 1 and 3 (the lower left panel of Figure 6). Figures 7 and 8 provide the analogous plots for the second subsample. Here the biggest problems come from the fact that the equations one would want to use to forecast the spread in weeks 3 and 4 are quite different from those found for weeks 1 and 2 (see the right-hand column of Figure 7). These

considerations suggest looking at models that allow for more general seasonal variation than our baseline specification, which we explore in the next section.

## 5 Less restrictive seasonal models.

### 5.1 Structural estimation.

Here we consider a system in which the dynamic process followed by the factors is itself dependent on which week of the month we are looking at:

$$x_{t+1} = c_{j_t} + \rho_{j_t} x_t + \Sigma_{j_t} u_{t+1}.$$

If we hypothesize that the risk-pricing parameters also vary with the season,

$$\lambda_t = \lambda_{j_t} + \Lambda_{j_t} x_t$$

then the no-arbitrage conditions (9) and (10) generalize to

$$\beta'_n = \beta'_{n-1} \rho_{j(n)}^Q \tag{35}$$

$$\alpha_n = \alpha_{n-1} + \beta'_{n-1} c_{j(n)}^Q + (1/2) \beta'_{n-1} \Sigma_{j(n)} \Sigma'_{j(n)} \beta_{n-1}.$$

where which observation week  $j$  is associated with a given maturity  $n$  can be read off of Table 1 and where we have defined  $\rho_j^Q = \rho_j - \Lambda_j$  and  $c_j^Q = c_j - \lambda_j$ . Unfortunately, if all the parameters were allowed to vary with the week  $j$  in this way, the model would be unidentified. The reason

is that even if one hypothesizes different values of  $\rho_j^Q$  for different  $j$ , a generalization of the algebra in (47) still implies that  $\Gamma_j$  should be the same for all  $j$ :

$$\Gamma_j = H_2 \begin{bmatrix} \beta'_0 \\ \beta'_4 \\ \beta'_8 \end{bmatrix} \quad \text{for } j = 1, 2, 3, 4.$$

Since  $\beta'_4 = \beta'_0 \rho_3^Q \rho_2^Q \rho_1^Q \rho_4^Q$  and  $\beta'_8 = \beta'_4 \rho_3^Q \rho_2^Q \rho_1^Q \rho_4^Q$ , the only information available from the regressions in which  $y_{2t}$  is the dependent variable (24) is about the product  $\rho_3^Q \rho_2^Q \rho_1^Q \rho_4^Q$ , which does not allow identification of the individual terms. In the next subsection we report estimates for a system in which although  $c_j$ ,  $\rho_j$ ,  $\lambda_j$ , and  $\Lambda_j$  all vary with  $j$ , the differences  $c^Q = c_j - \lambda_j$  and  $\rho^Q = \rho_j - \Lambda_j$  do not. For this system, the flexibility of the  $c_j$  and  $\rho_j$  parameters allows us to fit the unrestricted OLS values for  $\phi_j$  and  $\Phi_j$  perfectly. Details of the normalization and estimation for this less restrictive specification are reported in Appendices B and C.

## 5.2 Empirical results for the less restrictive seasonal model.

Empirical estimates for the parameters of the above system for each of the two subsamples are reported in Tables 5 and 6. In the first subsample, the main differences are that the specification allows the spread to become harder to forecast as the near contract approaches expiry (that is, the (2,2) element of  $\Sigma_j$  increases in  $j$ ) and the level and slope at the end of the month are less related to their values at expiry than is typical of the relation between  $y_{1t}$  and  $y_{1,t-1}$  at other times (that is, diagonal elements of  $\rho_j$  are smaller for  $j = 4$ ). Although implied values for  $\lambda$  and  $\Lambda$  are estimated with much less precision, the overall conclusion that

individual elements are small and statistically insignificant applies across individual weeks as well.

For the second subsample (Table 6), the dependence of  $\Lambda_{12,j}$  on week  $j$  is very dramatic, with an average value of  $-0.78$  for  $j = 1, 2$ , or  $3$  but an estimated value of  $+0.46$  for  $j = 4$ . A high spread signals lower returns to the long position during weeks 1-3, but this effect completely disappears, and may even take on the opposite sign, during expiry week 4. This may be related to the strong weekly pattern to index-fund strategies. For example, to replicate the crude oil holdings of the Goldman Sachs Commodity Index, an index fund would be selling the  $k = 0$  contract and buying the  $k = 1$  contract during week  $j = 3$ . It is interesting that we also find strong weekly patterns in the pricing of risk in data since 2005, though trying to interpret those changes in detail is beyond the scope of this paper.

Although our more general specification can fit the unrestricted OLS estimates  $\hat{\phi}_j$  and  $\hat{\Phi}_j$  perfectly, it still imposes testable overidentifying restrictions on other parameters, essentially using the 3 parameters in  $\{\alpha_0, \xi_1, \xi_2\}$  to fit the 12 values for  $\{\hat{\gamma}_j, \hat{\Gamma}_j\}_{j=1}^4$ . The resulting  $\chi^2(9)$  MCS test statistic for the first subsample is 13.86, which with a  $p$ -value of 0.13 is consistent with the null hypothesis that the model has adequately captured all the week-to-week variations in parameters. The second subsample ( $\chi^2(9) = 13.25$ ,  $p = 0.15$ ) also passes this specification test.

## 6 Conclusions.

In this paper, we studied the interaction between hedging demands from commercial producers or financial investors and risk aversion on the part of the arbitrageurs who are persuaded to be the hedgers' counterparties. We demonstrated that this interaction can produce an affine factor structure for the log prices of futures contracts in which expected returns depend on the arbitrageurs' net exposure to nondiversifiable risk. We developed new algorithms for estimation and diagnostic tools for testing this class of models appropriate for an unbalanced data set in which the duration of observed contracts changes with each observation.

Prior to 2005, we found that someone who consistently took the long side of nearby oil futures contracts received positive compensation on average, with relatively modest variation of this risk premium over time, consistent with the interpretation that the primary source of this premium was hedging by commercial producers. However, we uncovered significant changes in the pricing of risk after the volume of trading in these contracts increased significantly in 2005. The expected compensation from a long position is lower on average but much more volatile in the recent data, often significantly negative when the futures curve slopes upward. We suggest that increased participation by financial investors in oil futures markets may have been a factor in changing the nature of risk premia in crude oil futures contracts.

## References

- Acharya, Viral V., Lars A. Lochstoer, and Tarun Ramadorai (2010) “Limits to arbitrage and hedging: Evidence from commodity markets” Working paper, London Business School.
- Alquist, Ron, and Olivier Gervais (2011) “The role of financial speculation in driving the price of crude oil” Working paper, Bank of Canada.
- Alquist, Ron, and Lutz Kilian (2010) “What do we learn from the price of crude oil futures?” *Journal of Applied Econometrics* 25, 539–573.
- Ang, Andrew, and Monika Piazzesi (2003) “A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables” *Journal of Monetary Economics* 50, 745–787.
- Bauer, Michael D., Glenn D. Rudebusch, and Jing Wu (2011) “Unbiased estimation of dynamic term structure models” Working paper.
- Bessembinder, Hendrik (1992) “Systemic risk, hedging pressure, and risk premiums in futures markets” *Review of Financial Studies* 5, 637–667.
- Buyuksahin, Bahattin, and Michel A. Robe (2011) “Does ‘paper oil’ matter?” Working paper, American University.
- Carter, Colin A., Gordon C. Rausser, and Andrew Schmitz (1983) “Efficient asset portfolios and the theory of normal backwardation” *Journal of Political Economy* 91, 319–331.

- Casassus, Jaime, and Pierre Collin-Dufresne (2006) “Stochastic convenience yield implied from commodity futures and interest rates” *Journal of Finance* 60, 2283–2331.
- Chang, Eric C. (1985) “Returns to speculators and the theory of normal backwardation” *Journal of Finance* 40, 193–208.
- Chen, Ren-RAW, and Louis Scott (1993) “Maximum likelihood estimation for a multifactor equilibrium model of the term structure of interest rates” *The Journal of Fixed Income* 3, 14–31.
- Cochrane, John H., and Monika Piazzesi (2009) “Decomposing the yield curve” AFA 2010 Atlanta Meetings Paper.
- Cortazar, Gonzalo, and Lorenzo Naranjo (2006) “An n-factor gaussian model of oil futures prices” *Journal of Futures Markets* 26, 243–268.
- Dai, Qiang, and Kenneth J. Singleton (2002) “Expectation puzzles, time-varying risk premia, and affine models of the term structure” *Journal of Financial Economics* 63, 415–441.
- De Roon, Frans, Theo Nijman, and Chris Veld (2000) “Hedging pressure effects in futures markets” *Journal of Finance* 55, 1437–1456.
- Duffee, Gregory R. (2002) “Term premia and interest rate forecasts in affine models” *The Journal of Finance* 57, 405–443.
- Duffie, Darrell (1992) *Dynamic Asset Pricing Theory* Princeton University Press, Princeton, NJ.

- Duffie, Darrell, and Rui Kan (1996) “A yield-factor model of interest rates” *Mathematical Finance* 6, 379–406.
- Hamilton, James D. (1994) *Time Series Analysis* Princeton University Press, Princeton, New Jersey.
- Hamilton, James D. (2009) “Causes and consequences of the oil shock of 2007-08” *Brookings Papers on Economic Activity* Spring 2009, 215–259.
- Hamilton, James D., and Jing Cynthia Wu (2010) “Identification and estimation of gaussian affine term structure models” Working paper, University of California, San Diego.
- Hamilton, James D., and Jing Cynthia Wu (forthcominga) “The effectiveness of alternative monetary policy tools in a zero lower bound environment” *Journal of Money, Credit & Banking*.
- Hamilton, James D., and Jing Cynthia Wu (forthcomingb) “Testable implications of affine term structure models” *Journal of Econometrics*.
- Irwin, Scott H., and Dwight R. Sanders (2011) “Testing the masters hypothesis in commodity futures markets” Working paper, University of Illinois.
- Joslin, Scott, Kenneth J. Singleton, and Haoxiang Zhu (forthcoming) “A new perspective on gaussian dynamic term structure models” *Review of Financial Studies*.
- Keynes, John M. (1930) *A Treatise on Money, Volume 2* Macmillan, London.
- Mou, Yiqun (2010) “Limits to arbitrage and commodity index investment: Front-running the goldman roll” Working paper, Columbia University.

- Schwartz, Eduardo S. (1997) “The stochastic behavior of commodity prices: Implications for valuation and hedging” *Journal of Finance* 52, 923–973.
- Schwartz, Eduardo S., and James E. Smith (2000) “Short-term variations and long-term dynamics in commodity prices” *Management Science* 46, 893–911.
- Singleton, Kenneth J. (2011) “Investor flows and the 2008 boom/bust in oil prices” Working paper, Stanford University.
- Tang, Ke, and Wei Xiong (2011) “Index investment and the financialization of commodities” Working paper, Princeton University.
- Vasicek, Oldrich (1977) “An equilibrium characterization of the term structure” *Journal of Financial Economics* 5, 177–188.
- Vayanos, Dimitri, and Jean-Luc Vila (2009) “A preferred-habitat model of the term structure of interest rates” Working paper, London School of Economics.

## 7 Appendix A. Approximations to portfolio mean and variance.

Note

$$\log(F_{n-1,t+1}/F_{nt}) \sim N(\mu_{n-1,t}, \sigma_{n-1}^2)$$

$$\mu_{n-1,t} = \alpha_{n-1} + \beta'_{n-1}(c + \rho x_t) - \alpha_n - \beta'_n x_t \quad (36)$$

$$\sigma_{n-1}^2 = \beta'_{n-1} \Sigma \Sigma' \beta_{n-1} \quad (37)$$

$$E_t(F_{n-1,t+1}/F_{nt}) = \exp(\mu_{n-1,t} + \sigma_{n-1}^2/2).$$

A first-order Taylor expansion around  $\mu_{n-1,t} = \sigma_{n-1}^2 = 0$  gives

$$E_t(F_{n-1,t+1}/F_{nt}) \approx 1 + \mu_{n-1,t} + \sigma_{n-1}^2/2$$

from which

$$\begin{aligned} E_t(W_{t+1}) &= \sum_{n=1}^N z_{nt} [E_t(F_{n-1,t+1}/F_{nt}) - 1] \\ &\approx \sum_{n=1}^N z_{nt} [\mu_{n-1,t} + \sigma_{n-1}^2/2]. \end{aligned} \quad (38)$$

Substituting (36) and (37) into (38) gives (4).

Likewise, if

$$\begin{bmatrix} \log X_i \\ \log X_j \end{bmatrix} \sim N \left( \begin{bmatrix} \mu_i \\ \mu_j \end{bmatrix}, \begin{bmatrix} \sigma_{ii} & \sigma_{ij} \\ \sigma_{ji} & \sigma_{jj} \end{bmatrix} \right),$$

then

$$\text{Cov}(X_i, X_j) = \exp [(\mu_i + \mu_j) + (\sigma_{ii} + \sigma_{jj})/2] [\exp(\sigma_{ij}) - 1].$$

A first-order Taylor expansion around  $\mu_i = \mu_j = \sigma_{ii} = \sigma_{jj} = \sigma_{ij} = 0$  gives

$$\text{Cov}(X_i, X_j) \approx \sigma_{ij}.$$

Since

$$\text{Cov}_t(f_{n-1,t+1} - f_{nt}, f_{\ell-1,t+1} - f_{\ell t}) = \beta'_{n-1} \Sigma \Sigma' \beta_{\ell-1},$$

we have

$$\begin{aligned} \text{Var}_t(W_{t+1}) &= \sum_{n=1}^N \sum_{\ell=1}^N z_{nt} z_{\ell t} \text{Cov}_t(F_{n-1,t+1}/F_{nt}, F_{\ell-1,t+1}/F_{\ell t}) \\ &\approx \sum_{n=1}^N \sum_{\ell=1}^N z_{nt} z_{\ell t} \beta'_{n-1} \Sigma \Sigma' \beta_{\ell-1} \\ &= \left( \sum_{n=1}^N z_{nt} \beta'_{n-1} \right) \Sigma \Sigma' \left( \sum_{\ell=1}^N z_{\ell t} \beta_{\ell-1} \right). \end{aligned}$$

## 8 Appendix B. Normalization.

**Baseline model.** Let  $\rho^Q = \rho - \Lambda$ , and notice from (9) that

$$\beta_n = (\rho^Q)^n \beta_0. \tag{39}$$

We will parameterize this in terms of  $\xi = (\xi_1, \xi_2)'$  where  $\xi_i$  denotes an eigenvalue of  $\rho^{Q'}$ . We could calculate the following matrix as a function of those eigenvalues:

$$K(\xi) = \begin{matrix} & & & \\ & & & \\ & & & \\ (2 \times 3) & \begin{bmatrix} \xi_1^0 & \xi_1^4 & \xi_1^8 \\ \xi_2^0 & \xi_2^4 & \xi_2^8 \end{bmatrix} & & \end{matrix}. \quad (40)$$

The claim is that if we specify

$$\rho^{Q'} = \begin{matrix} & & & \\ & & & \\ & & & \\ (2 \times 2) & \begin{bmatrix} K(\xi) & H'_1 \\ (2 \times 3) & (3 \times 2) \end{bmatrix}^{-1} \begin{bmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{bmatrix} \begin{bmatrix} K(\xi) & H'_1 \\ (2 \times 3) & (3 \times 2) \end{bmatrix} & & \end{matrix} \quad (41)$$

$$\beta_0 = \begin{matrix} & & & \\ & & & \\ & & & \\ (2 \times 1) & \begin{bmatrix} K(\xi) & H'_1 \\ (2 \times 3) & (3 \times 2) \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} & & \end{matrix}, \quad (42)$$

then (32), the desired condition for  $\beta_n$ , would be satisfied. To prove this, observe from (39)

that

$$\begin{aligned} \beta_n &= [K(\xi)H'_1]^{-1} \begin{bmatrix} \xi_1^n & 0 \\ 0 & \xi_2^n \end{bmatrix} [K(\xi)H'_1] [K(\xi)H'_1]^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= [K(\xi)H'_1]^{-1} \begin{bmatrix} \xi_1^n \\ \xi_2^n \end{bmatrix} \end{aligned} \quad (43)$$

so that

$$H_1 \begin{bmatrix} \beta'_0 \\ \beta'_4 \\ \beta'_8 \end{bmatrix} = H_1 \begin{bmatrix} \xi_1^0 & \xi_2^0 \\ \xi_1^4 & \xi_2^4 \\ \xi_1^8 & \xi_2^8 \end{bmatrix} [H_1 K(\xi)']^{-1}. \quad (44)$$

Substituting (40) into (44) produces (32), as claimed. Thus if we know  $\xi$ , we can use (40) and (43) to calculate the value of  $\beta_n$  for any  $n$  as well as  $\rho^Q$  from (41).

To achieve the separate condition (33) on  $\alpha_n$ , notice from (10) that

$$\begin{aligned} \alpha_n &= \alpha_0 + (\beta'_{n-1} + \beta'_{n-2} + \cdots + \beta'_0)c^Q \\ &\quad + (1/2)(\beta'_{n-1}\Sigma\Sigma'\beta_{n-1} + \beta'_{n-2}\Sigma\Sigma'\beta_{n-2} + \cdots + \beta'_0\Sigma\Sigma'\beta_0). \end{aligned} \quad (45)$$

Define

$$\begin{aligned} \zeta_n(\xi) &= \beta'_{n-1} + \beta'_{n-2} + \cdots + \beta'_0 \\ &\quad \text{(1}\times\text{2)} \\ \psi_n(\xi, \alpha_0, \Sigma) &= \alpha_0 + (1/2)(\beta'_{n-1}\Sigma\Sigma'\beta_{n-1} + \beta'_{n-2}\Sigma\Sigma'\beta_{n-2} + \cdots + \beta'_0\Sigma\Sigma'\beta_0) \\ &\quad \text{(1}\times\text{1)} \end{aligned}$$

so that (45) can be written

$$\alpha_n = \zeta_n(\xi)c^Q + \psi_n(\xi, \alpha_0, \Sigma)$$

where for  $n = 0$  we have  $\zeta_0(\xi) = 0$  and  $\psi_0(\xi, \alpha_0, \Sigma) = \alpha_0$ . We claim that if we choose

$$c^Q = - \left( H_1 \begin{bmatrix} \zeta_0(\xi) \\ \zeta_4(\xi) \\ \zeta_8(\xi) \end{bmatrix} \right)^{-1} \left( H_1 \begin{bmatrix} \psi_0(\xi, \alpha_0, \Sigma) \\ \psi_4(\xi, \alpha_0, \Sigma) \\ \psi_8(\xi, \alpha_0, \Sigma) \end{bmatrix} \right), \quad (46)$$

then (33) would be satisfied. This is demonstrated as follows:

$$\begin{aligned}
H_1 \begin{bmatrix} \alpha_0 \\ \alpha_4 \\ \alpha_8 \end{bmatrix} &= H_1 \begin{bmatrix} \zeta_0(\xi)c^Q + \psi_0(\xi, \alpha_0, \Sigma) \\ \zeta_4(\xi)c^Q + \psi_4(\xi, \alpha_0, \Sigma) \\ \zeta_8(\xi)c^Q + \psi_8(\xi, \alpha_0, \Sigma) \end{bmatrix} \\
&= -H_1 \begin{bmatrix} \psi_0(\xi, \alpha_0, \Sigma) \\ \psi_4(\xi, \alpha_0, \Sigma) \\ \psi_8(\xi, \alpha_0, \Sigma) \end{bmatrix} + H_1 \begin{bmatrix} \psi_0(\xi, \alpha_0, \Sigma) \\ \psi_4(\xi, \alpha_0, \Sigma) \\ \psi_8(\xi, \alpha_0, \Sigma) \end{bmatrix} \\
&= 0.
\end{aligned}$$

Thus if we know  $\xi$ ,  $\alpha_0$ , and  $\Sigma$ , we can use (45) and (46) to calculate the value of  $\alpha_n$  for any  $n$ . Also  $\xi$ ,  $\alpha_0$ , and  $\Sigma$  allow calculation of  $c^Q$  from (46).

**Seasonal model.** Just as in the baseline model, we let  $(\xi_1, \xi_2)$  denote the ordered eigenvalues of  $\rho^Q$  and write

$$\beta'_n = \begin{bmatrix} \xi_1^n & \xi_2^n \end{bmatrix} [H_1 K(\xi)']^{-1}$$

which achieves the normalization (32). Likewise for  $\alpha_n$  we again use (46) where now

$$\zeta_n(\xi) = \beta'_{n-1} + \beta'_{n-2} + \cdots + \beta'_0$$

$$\psi_n(\xi, \alpha_0, \Sigma) = \alpha_0 + (1/2)(\beta'_{n-1}\Sigma_{j(n)}\Sigma'_{j(n)}\beta_{n-1} + \beta'_{n-2}\Sigma_{j(n-1)}\Sigma'_{j(n-1)}\beta_{n-2} + \cdots + \beta'_0\Sigma_{j(1)}\Sigma'_{j(1)}\beta_0).$$

## 9 Appendix C. Mapping from structural to reduced-form parameters.

**Baseline model.** Expressions involving  $B_{1j}$  in (21) can be simplified by noting from (39) and (32) that

$$B_{1j} = H_1 \begin{bmatrix} \beta'_{4-j} \\ \beta'_{8-j} \\ \beta'_{12-j} \end{bmatrix} = H_1 \begin{bmatrix} \beta'_0 \\ \beta'_4 \\ \beta'_8 \end{bmatrix} (\rho^Q)^{4-j} = (\rho^Q)^{4-j}.$$

Thus for example (25) simplifies to

$$\begin{aligned} \Gamma_j &= H_2 \begin{bmatrix} \beta'_{4-j} \\ \beta'_{8-j} \\ \beta'_{12-j} \end{bmatrix} \left( H_1 \begin{bmatrix} \beta'_{4-j} \\ \beta'_{8-j} \\ \beta'_{12-j} \end{bmatrix} \right)^{-1} \\ &= H_2 \begin{bmatrix} \beta'_0 \\ \beta'_4 \\ \beta'_8 \end{bmatrix} (\rho^Q)^{4-j} \left[ (\rho^Q)^{4-j} \right]^{-1} \\ &= H_2 \begin{bmatrix} \beta'_0 \\ \beta'_4 \\ \beta'_8 \end{bmatrix} \quad \text{for } j = 1, 2, 3, 4. \end{aligned} \tag{47}$$

The population magnitudes corresponding to the other reduced-form OLS coefficients are as follows:

$$\Omega_j = (\rho^Q)^{4-j} \Sigma \Sigma' (\rho^{Q'})^{4-j} \quad \text{for } j = 1, 2, 3, 4$$

$$\Phi_1 = (\rho^Q)^3 \rho$$

$$\Phi_j = (\rho^Q)^{4-j} \rho \left[ (\rho^Q)^{-1} \right]^{4-j+1} \quad \text{for } j = 2, 3, 4$$

$$\phi_1 = H_1 \begin{bmatrix} \alpha_3 \\ \alpha_7 \\ \alpha_{11} \end{bmatrix} + H_1 \begin{bmatrix} \beta'_3 \\ \beta'_7 \\ \beta'_{11} \end{bmatrix} c - \Phi_1 H_1 \begin{bmatrix} \alpha_0 \\ \alpha_4 \\ \alpha_8 \end{bmatrix}$$

$$\phi_j = H_1 \begin{bmatrix} \alpha_{4-j} \\ \alpha_{8-j} \\ \alpha_{12-j} \end{bmatrix} + H_1 \begin{bmatrix} \beta'_{4-j} \\ \beta'_{8-j} \\ \beta'_{12-j} \end{bmatrix} c - \Phi_j H_1 \begin{bmatrix} \alpha_{5-j} \\ \alpha_{9-j} \\ \alpha_{13-j} \end{bmatrix} \quad \text{for } j = 2, 3, 4$$

$$\gamma_j = H_2 \begin{bmatrix} \alpha_{4-j} \\ \alpha_{8-j} \\ \alpha_{12-j} \end{bmatrix} - \Gamma_j H_1 \begin{bmatrix} \alpha_{4-j} \\ \alpha_{8-j} \\ \alpha_{12-j} \end{bmatrix} \quad \text{for } j = 1, 2, 3, 4.$$

Given the scalars  $\{\xi_1, \xi_2\}$  (corresponding to the eigenvalues of  $\rho^Q = \rho - \Lambda$ ), we can calculate  $\beta_n$  from (43) and (40). These  $\beta_n$  give us predicted values for  $\{\Gamma_j\}_{j=1}^4$ , and the  $\beta_n$  along with  $\Sigma$  give predicted values for  $\{\Omega_j\}_{j=1}^4$ . Note  $\{\xi_1, \xi_2\}$  also gives us  $\rho^Q$ , and this plus  $\rho$  gives predicted values for  $\{\Phi_j\}_{j=1}^4$ . From  $\beta_n$ ,  $\Sigma$ , and  $\alpha_0$  we can calculate  $c^Q$  from (46) and  $\alpha_n$  from (45). Using these along with  $c$  we then obtain the predicted values for  $\{\phi_j\}_{j=1}^4$  and  $\{\gamma_j\}_{j=1}^4$ .

The information matrix for the OLS estimates  $\hat{\pi} = (\hat{\pi}'_{\Phi}, \hat{\pi}'_{\Omega}, \hat{\pi}'_{\Gamma}, \hat{\pi}'_{\sigma})'$  is given by

$$\hat{R}_\pi = \begin{bmatrix} \hat{R}_\Phi & 0 & 0 & 0 \\ 0 & \hat{R}_\Omega & 0 & 0 \\ 0 & 0 & \hat{R}_\Gamma & 0 \\ 0 & 0 & 0 & \hat{R}_\sigma \end{bmatrix} \quad (48)$$

$$\hat{R}_\Phi = \begin{bmatrix} \hat{R}_{\Phi 1} & 0 & 0 & 0 \\ 0 & \hat{R}_{\Phi 2} & 0 & 0 \\ 0 & 0 & \hat{R}_{\Phi 3} & 0 \\ 0 & 0 & 0 & \hat{R}_{\Phi 4} \end{bmatrix}$$

$$\hat{R}_{\Phi j} = \hat{\Omega}_j^{-1} \otimes \mathcal{T}^{-1} \sum_{\tau=1}^{\mathcal{T}} x_{1,j,\tau}^\dagger x_{1,j,\tau}^\dagger$$

$$\hat{R}_\Omega = \begin{bmatrix} \hat{R}_{\Omega 1} & 0 & 0 & 0 \\ 0 & \hat{R}_{\Omega 2} & 0 & 0 \\ 0 & 0 & \hat{R}_{\Omega 3} & 0 \\ 0 & 0 & 0 & \hat{R}_{\Omega 4} \end{bmatrix}$$

$$\hat{R}_{\Omega j} = (1/2) D_2' \left( \hat{\Omega}_j^{-1} \otimes \hat{\Omega}_j^{-1} \right) D_2$$

$$D_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{R}_\Gamma = \begin{bmatrix} \hat{R}_{\Gamma 1} & 0 & 0 & 0 \\ 0 & \hat{R}_{\Gamma 2} & 0 & 0 \\ 0 & 0 & \hat{R}_{\Gamma 3} & 0 \\ 0 & 0 & 0 & \hat{R}_{\Gamma 4} \end{bmatrix}$$

$$\hat{R}_{\Gamma j} = \hat{\sigma}_{e_j}^{-2} \left( T^{-1} \sum_{\tau=1}^T x_{2,j,\tau}^\dagger x_{2,j,\tau}^{\dagger'} \right)$$

$$\hat{R}_\sigma = \begin{bmatrix} (1/2)\hat{\sigma}_{e1}^{-4} & 0 & 0 & 0 \\ 0 & (1/2)\hat{\sigma}_{e2}^{-4} & 0 & 0 \\ 0 & 0 & (1/2)\hat{\sigma}_{e3}^{-4} & 0 \\ 0 & 0 & 0 & (1/2)\hat{\sigma}_{e4}^{-4} \end{bmatrix}$$

where  $D_2$  is the duplication matrix satisfying  $D_2 \text{vech}(\Omega) = \text{vec}(\Omega)$  for  $2 \times 2$  symmetric matrix  $\Omega$ . Note that for all models considered the MCS estimate of  $\sigma_{e_j}$  is always equal to the unconstrained MLE  $\hat{\sigma}_{e_j}$  and so contributes 0 to the weighted objective function.

**Unrestricted seasonal model.** For this model the specifications for  $\gamma_j$  and  $\Gamma_j$  are the same as in the baseline model, while the expressions for the other parameters become

$$\Omega_1 = (\rho^Q)^3 \Sigma_4 \Sigma_4' (\rho^{Q'})^3$$

$$\Omega_j = (\rho^Q)^{4-j} \Sigma_{j-1} \Sigma_{j-1}' (\rho^{Q'})^{4-j} \quad \text{for } j = 2, 3, 4$$

$$\Phi_1 = (\rho^Q)^3 \rho_4$$

$$\Phi_j = (\rho^Q)^{4-j} \rho_{j-1} \left[ (\rho^Q)^{-1} \right]^{4-j+1} \quad \text{for } j = 2, 3, 4 \quad (49)$$

$$\phi_1 = H_1 \begin{bmatrix} \alpha_3 \\ \alpha_7 \\ \alpha_{11} \end{bmatrix} + H_1 \begin{bmatrix} \beta'_3 \\ \beta'_7 \\ \beta'_{11} \end{bmatrix} c_4 - \Phi_1 H_1 \begin{bmatrix} \alpha_0 \\ \alpha_4 \\ \alpha_8 \end{bmatrix}$$

$$\phi_j = H_1 \begin{bmatrix} \alpha_{4-j} \\ \alpha_{8-j} \\ \alpha_{12-j} \end{bmatrix} + H_1 \begin{bmatrix} \beta'_{4-j} \\ \beta'_{8-j} \\ \beta'_{12-j} \end{bmatrix} c_{j-1} - \Phi_j H_1 \begin{bmatrix} \alpha_{5-j} \\ \alpha_{9-j} \\ \alpha_{13-j} \end{bmatrix} \quad \text{for } j = 2, 3, 4.$$

Minimum-chi-square estimation in this case is achieved by first choosing  $\{\xi, \alpha_0, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4\}$  so as to minimize the distance from the OLS estimates  $\{\hat{\gamma}_j, \hat{\Gamma}_j, \hat{\Omega}_j\}_{j=1}^4$ . From  $\xi$  we can then calculate  $\rho^Q$ , with which we can obtain  $\rho_j$  analytically from (49) in order to fit these OLS coefficients perfectly. The values  $c_{j-1}$  are likewise obtained analytically from  $\hat{\phi}_j$ .

Table 1: Weekly durations associated with monthly contracts at different points in time

$j$	$k = 0$	$k = 1$	$k = 2$
1	3	7	11
2	2	6	10
3	1	5	9
4	0	4	8

For specified week of the month  $j$  and months until the contract expires  $k$ , table entry indicates weeks  $n$  remaining until expiry.

Table 2: Pre-2005 parameter estimates for baseline model

Estimated parameters			Implied parameters		
$c$	0.0102 (0.0209)	0.0016 (0.0029)	$\lambda$	0.0104 (0.0209)	-0.0061* (0.0029)
$\rho$	0.9974* (0.0067)	0.1839* (0.0918)	$\Lambda$	-0.0023 (0.0067)	-0.0591 (0.0918)
	-0.0006 (0.0009)	0.9301* (0.0134)		0.0020* (0.0009)	-0.0050 (0.0136)
$\xi$	0.9876* (0.0010)	0.9473* (0.0044)	$\lambda + \Lambda \bar{x}$	0.0037* (0.0018)	3.53e-005 (2.63e-004)
$\Sigma$	0.0449* (0.0012)	0 (0.0002)			
	-0.0038* (0.0002)	0.0047* (0.0001)			
$\alpha_0$	0.0357* (0.0007)				
$\pi_\sigma$	0.0099* (0.0005)	0.0081* (0.0004)			
	0.0105* (0.0006)	0.0201* (0.0011)			

Left panel: MCS estimates of elements of  $\theta$  for data from January 1990 through December 2004 (asymptotic standard errors in parentheses). Right panel: assorted magnitudes of interest implied by value of  $\hat{\theta}$  (asymptotic standard errors in parentheses). \* denotes statistically significant at the 5% level.

Table 3: Post-2005 parameter estimates for baseline model

Estimated parameters			Implied parameters		
$c$	0.1802*	0.0164*	$\lambda$	0.1813*	0.0179*
	(0.0574)	(0.0070)		(0.0574)	(0.0070)
$\rho$	0.9600*	-0.3487	$\Lambda$	-0.0400*	-0.5892*
	(0.0131)	(0.2018)		(0.0131)	(0.2018)
	-0.0035*	0.8629*		-0.0039*	-0.0311
	(0.0016)	(0.0241)		(0.0016)	(0.0243)
$\xi$	1.0010*	0.8931*	$\lambda + \Lambda \bar{x}$	0.0028	0.0009*
	(0.0001)	(0.0047)		(0.0026)	(0.0003)
$\Sigma$	0.0439*	0			
	(0.0019)				
	-0.0021*	0.0049*			
	(0.0003)	(0.0002)			
$\alpha_0$	-0.0086*				
	(0.0009)				
$\pi_\sigma$	0.0059*	0.0086*			
	(0.0005)	(0.0007)			
	0.0087*	0.0223*			
	(0.0007)	(0.0018)			

Left panel: MCS estimates of elements of  $\theta$  for data from January 2005 through June 2011 (asymptotic standard errors in parentheses). Right panel: assorted magnitudes of interest implied by value of  $\hat{\theta}$  (asymptotic standard errors in parentheses). \* denotes statistically significant at the 5% level.

Table 4:  $\chi^2$  specification test and breakdown by individual components

	$\chi^2$	$d.f.$	$p$ -value	$\pi_\Phi$	$\pi_\Omega$	$\pi_\Gamma$
Before 2005	86.57	36	4.73e-6	25.61	43.98	16.99
Since 2005	151.87	36	3.33e-16	120.01	17.22	14.64

$\chi^2$ : minimum value achieved for MCSE objective function.  $d.f.$ : degrees of freedom.  $p$ -value: probability of observing  $\chi^2(d.f.)$  value this large. Last 3 columns: contribution to  $\chi^2$  of individual parameter blocks.

Table 5: Pre-2005 parameter estimates for seasonal model

Estimated parameters					Implied parameters				
$c^{base}$	0.0102	0.0016			$\lambda^{base}$	0.0104	-0.0061*		
$c_1$	0.0042	0.0005			$\lambda_1$	0.0044	-0.0068		
	(0.0467)	(0.0056)				(0.0467)	(0.0055)		
$c_2$	-0.0359	0.0023			$\lambda_2$	-0.0356	-0.0050		
	(0.0422)	(0.0066)				(0.0422)	(0.0065)		
$c_3$	-0.0054	0.0053			$\lambda_3$	-0.0051	-0.0020		
	(0.0414)	(0.0055)				(0.0414)	(0.0054)		
$c_4$	0.0871*	-0.0003			$\lambda_4$	0.0874*	-0.0077		
	(0.0422)	(0.0073)				(0.0421)	(0.0079)		
$\rho^{base}$	0.9974*	0.1839*	-0.0006	0.9301*	$\Lambda^{base}$	-0.0023	-0.0591	0.0020*	-0.0050
$\rho_1$	0.9992*	0.2166	-0.0003	0.9369*	$\Lambda_1$	-0.0005	-0.0271	0.0022	-0.0061
	(0.0152)	(0.2386)	(0.0018)	(0.0285)		(0.0152)	(0.2386)	(0.0018)	(0.0281)
$\rho_2$	1.0137*	0.0371	-0.0010	0.9825*	$\Lambda_2$	0.0139	-0.2067	0.0015	0.0395
	(0.0138)	(0.2190)	(0.0022)	(0.0343)		(0.0138)	(0.2190)	(0.0021)	(0.0339)
$\rho_3$	1.0011*	0.5076*	-0.0018	0.9310*	$\Lambda_3$	0.0014	0.2639	0.0007	-0.0120
	(0.0135)	(0.2061)	(0.0018)	(0.0274)		(0.0135)	(0.2061)	(0.0018)	(0.0270)
$\rho_4$	0.9726*	-0.0718	0.0000	0.8690*	$\Lambda_4$	-0.0272*	-0.3155	0.0025	-0.0740
	(0.0138)	(0.2063)	(0.0024)	(0.0350)		(0.0138)	(0.2063)	(0.0026)	(0.0375)
$\Sigma^{base}$	0.0449*	-0.0038*	0.0047*						
$\Sigma_1$	0.0491*	-0.0042*	0.0039*						
	(0.0026)	(0.0004)	(0.0002)						
$\Sigma_2$	0.0447*	-0.0046*	0.0052*						
	(0.0024)	(0.0005)	(0.0003)						
$\Sigma_3$	0.0447*	-0.0029*	0.0051*						
	(0.0024)	(0.0004)	(0.0003)						
$\Sigma_4$	0.0446*	-0.0035*	0.0059*						
	(0.0024)	(0.0005)	(0.0003)						
$\xi^{base}$	0.9876*	0.9473*							
$\xi$	0.9854*	0.9574*							
	(0.0029)	(0.0070)							
$\alpha_0^{base}$	0.0357*								
$\alpha_0$	0.0331*								
	(0.0051)								
$\pi_\sigma$	0.0099*	0.0081*	0.0105*	0.0201*					

Left panel: MCS estimates for elements of  $\theta$  (asymptotic standard errors in parentheses) for the unrestricted seasonal model, with estimates from baseline model also reported for comparison, with all estimates based on data from January 1990 through December 2004. Elements of matrices reported as first row, then second row. Right panel: values implied by the reported estimates of  $\theta$ . \* denotes statistically significant at the 5% level.

Table 6: Post-2005 parameter estimates for seasonal model

Estimated parameters					Implied parameters				
$c^{base}$	0.1802*	0.0164*			$\lambda^{base}$	0.1813*	0.0179*		
$c_1$	0.0361 (0.1528)	0.0106 (0.0169)			$\lambda_1$	0.0374 (0.1528)	0.0119 (0.0167)		
$c_2$	0.2252* (0.1094)	-0.0299* (0.0147)			$\lambda_2$	0.2264* (0.1094)	-0.0286 (0.0144)		
$c_3$	0.0508 (0.1127)	0.0443* (0.0120)			$\lambda_3$	0.0520 (0.1127)	0.0456* (0.0119)		
$c_4$	0.1852 (0.1081)	0.0102 (0.0175)			$\lambda_4$	0.1865 (0.1081)	0.0115 (0.0184)		
$\rho^{base}$	0.9600*	-0.3487	-0.0035*	0.8629*	$\Lambda^{base}$	-0.0400*	-0.5892*	-0.0039*	-0.0311
$\rho_1$	0.9918* (0.0346)	-0.4445 (0.5708)	-0.0021 (0.0038)	1.0026* (0.0627)	$\Lambda_1$	-0.0082 (0.0346)	-0.6847 (0.5708)	-0.0025 (0.0038)	0.1141 (0.0622)
$\rho_2$	0.9488* (0.0247)	-0.5659 (0.3810)	0.0066* (0.0033)	1.1369* (0.0507)	$\Lambda_2$	-0.0512* (0.0247)	-0.8061* (0.3811)	0.0062 (0.0033)	0.2484* (0.0500)
$\rho_3$	0.9895* (0.0257)	-0.6038 (0.3454)	-0.0097* (0.0027)	0.6759* (0.0367)	$\Lambda_3$	-0.0105 (0.0257)	-0.8440* (0.3454)	-0.0100* (0.0027)	-0.2126* (0.0364)
$\rho_4$	0.9592* (0.0245)	0.7020 (0.4244)	-0.0021 (0.0040)	0.8796* (0.0661)	$\Lambda_4$	-0.0409 (0.0245)	0.4618 (0.4244)	-0.0025 (0.0042)	-0.0089 (0.0684)
$\Sigma^{base}$	0.0439*	-0.0021*	0.0049*						
$\Sigma_1$	0.0579* (0.0047)	-0.0031* (0.0007)	0.0055* (0.0005)						
$\Sigma_2$	0.0423* (0.0035)	-0.0013* (0.0006)	0.0054* (0.0004)						
$\Sigma_3$	0.0456* (0.0037)	-0.0020* (0.0005)	0.0044* (0.0004)						
$\Sigma_4$	0.0417* (0.0034)	-0.0029* (0.0007)	0.0055* (0.0005)						
$\xi^{base}$	1.0010*	0.8931*							
$\xi$	1.0008* (0.0010)	0.8877* (0.0059)							
$\alpha_0^{base}$	-0.0086*								
$\alpha_0$	-0.0077 (0.0106)								
$\pi_\sigma$	0.0059*	0.0086*	0.0087*	0.0223*					

Left panel: MCS estimates for elements of  $\theta$  (asymptotic standard errors in parentheses) for the unrestricted seasonal model, with estimates from baseline model also reported for comparison, with all estimates based on data from January 2005 through June 2011. Elements of matrices reported as first row, then second row. Right panel: values implied by the reported estimates of  $\theta$ . \* denotes statistically significant at the 5% level.

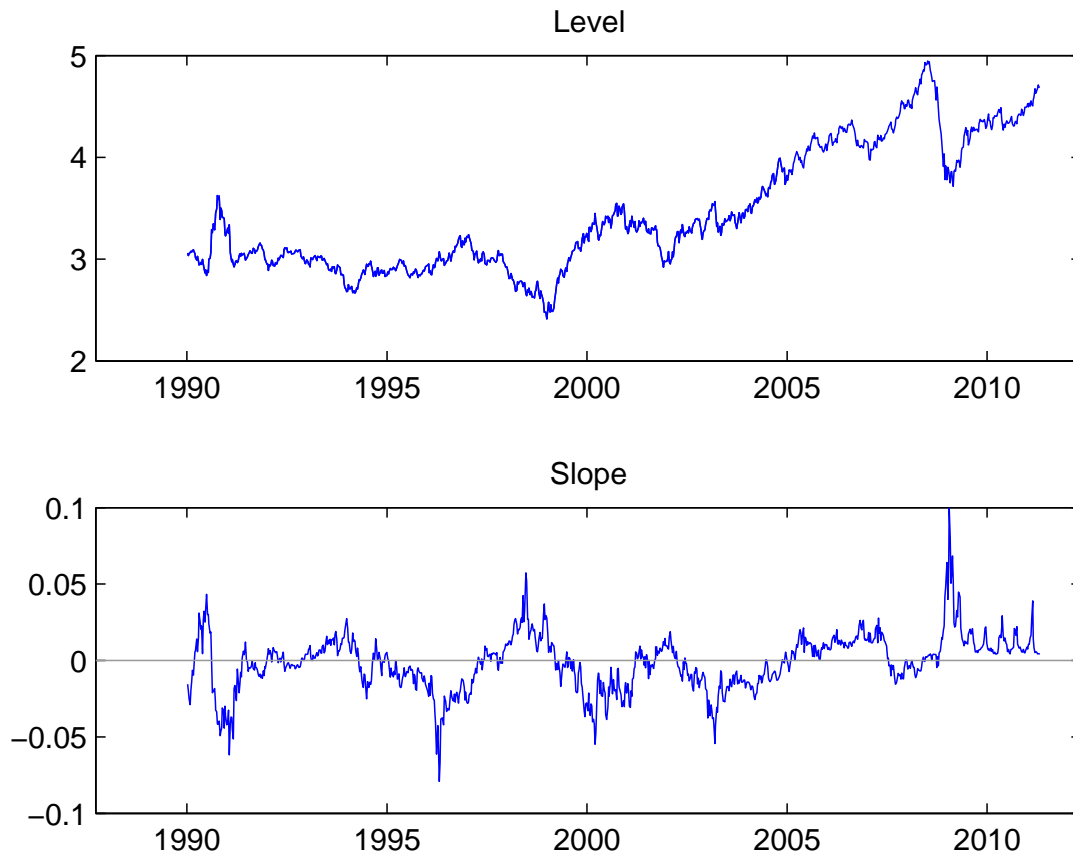


Figure 1: Data used in the analysis

Weekly observations (with specification of weeks as given in text), January 1990 to June 2011. Top panel: first element of  $y_{1t}$ . Bottom panel: second element of  $y_{1t}$ .

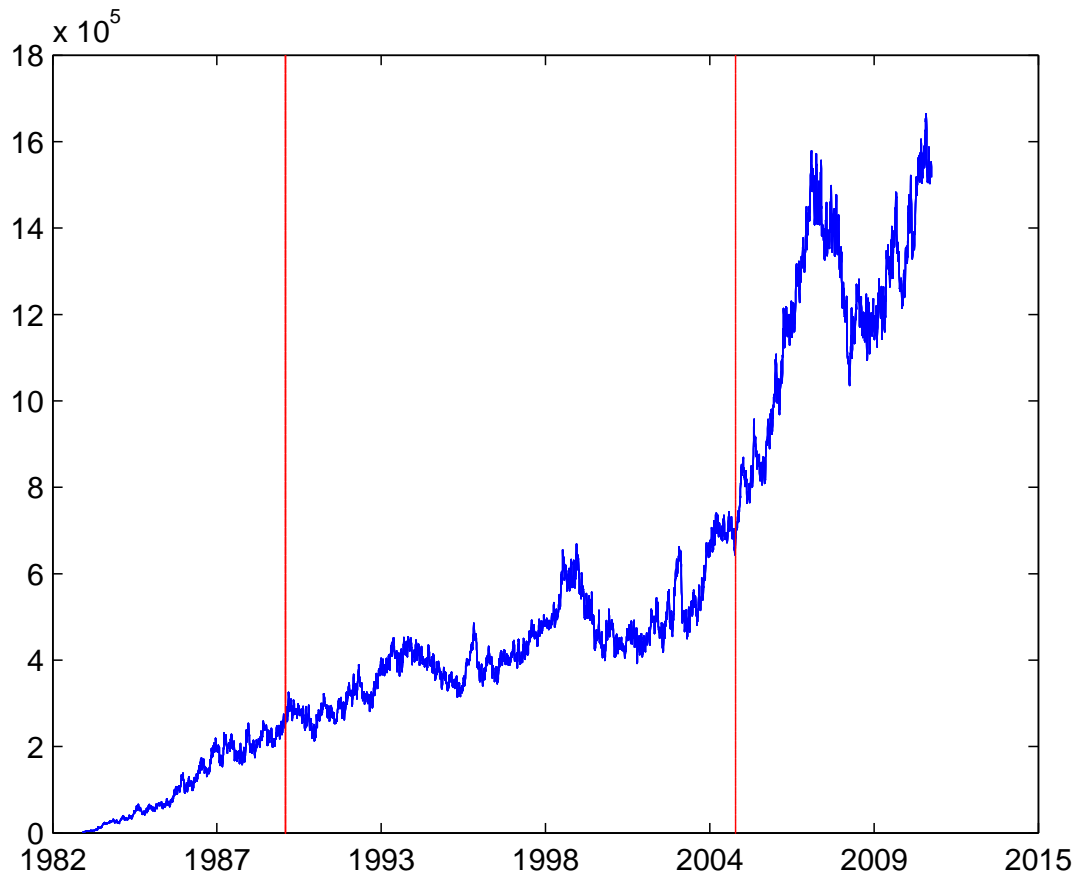


Figure 2: Open Interest

Total number of outstanding crude oil futures contracts of all maturities, daily observations, March 3, 1983 to July 17, 2011. Vertical lines drawn at January 1, 1990 and January 1, 2005.

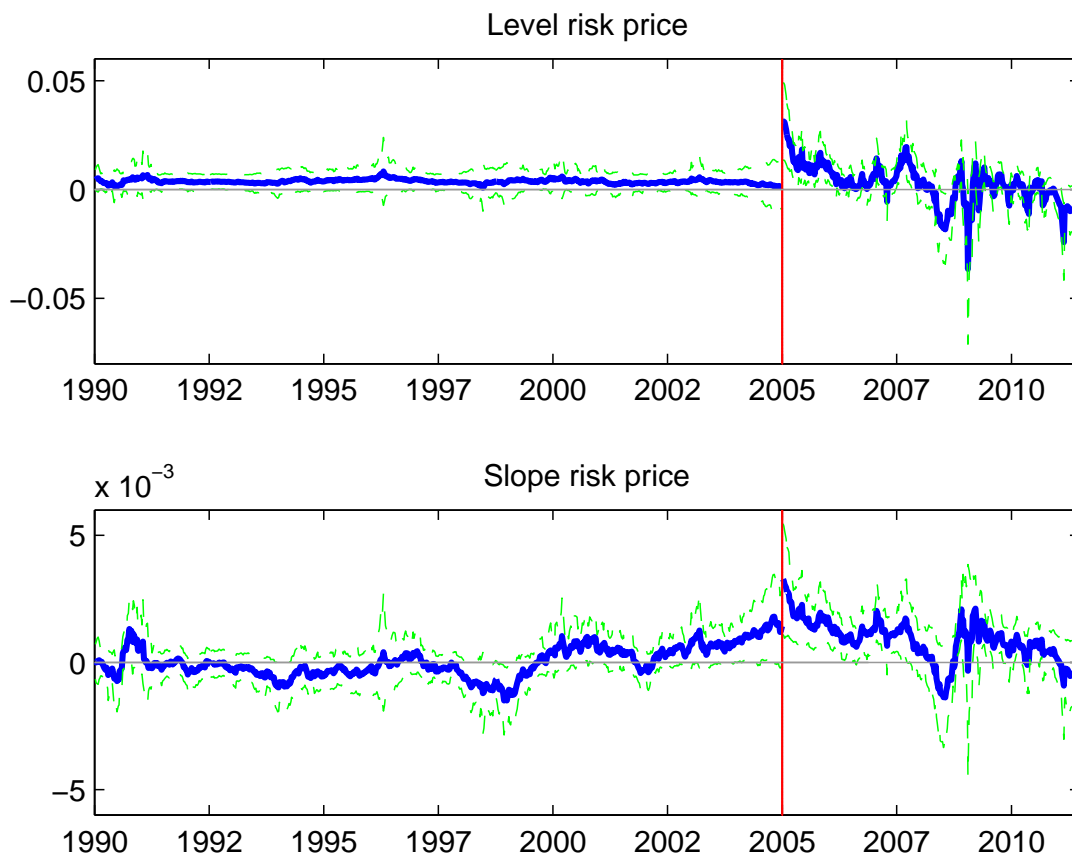


Figure 3: Prices of factor risk

Top panel: first element of  $\lambda + \Lambda x_t$  as estimated from baseline model, with sample split in 2005. Bottom panel: second element. Dashed lines indicate 95% confidence intervals.

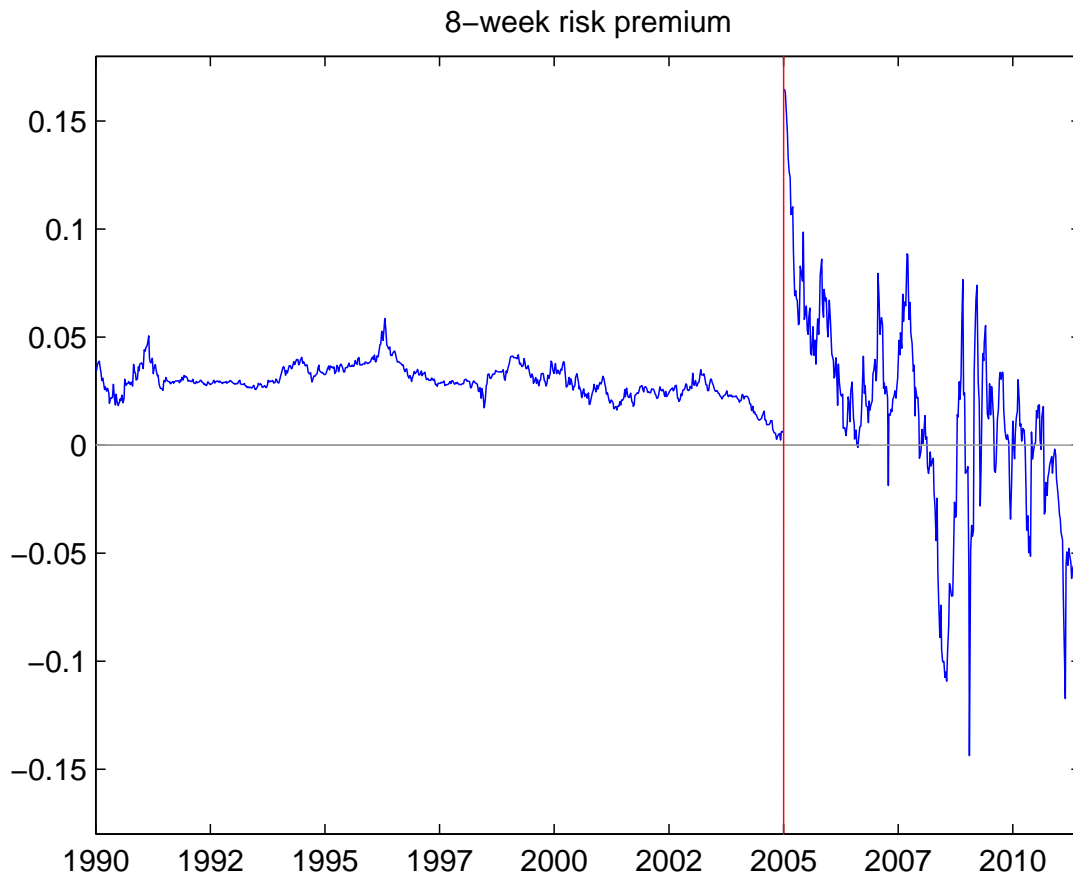


Figure 4: Risk premium on 8-week futures contract  
Plot of  $\tilde{f}_{8t} - f_{8t}$  as estimated from the baseline model with sample split in 2005.

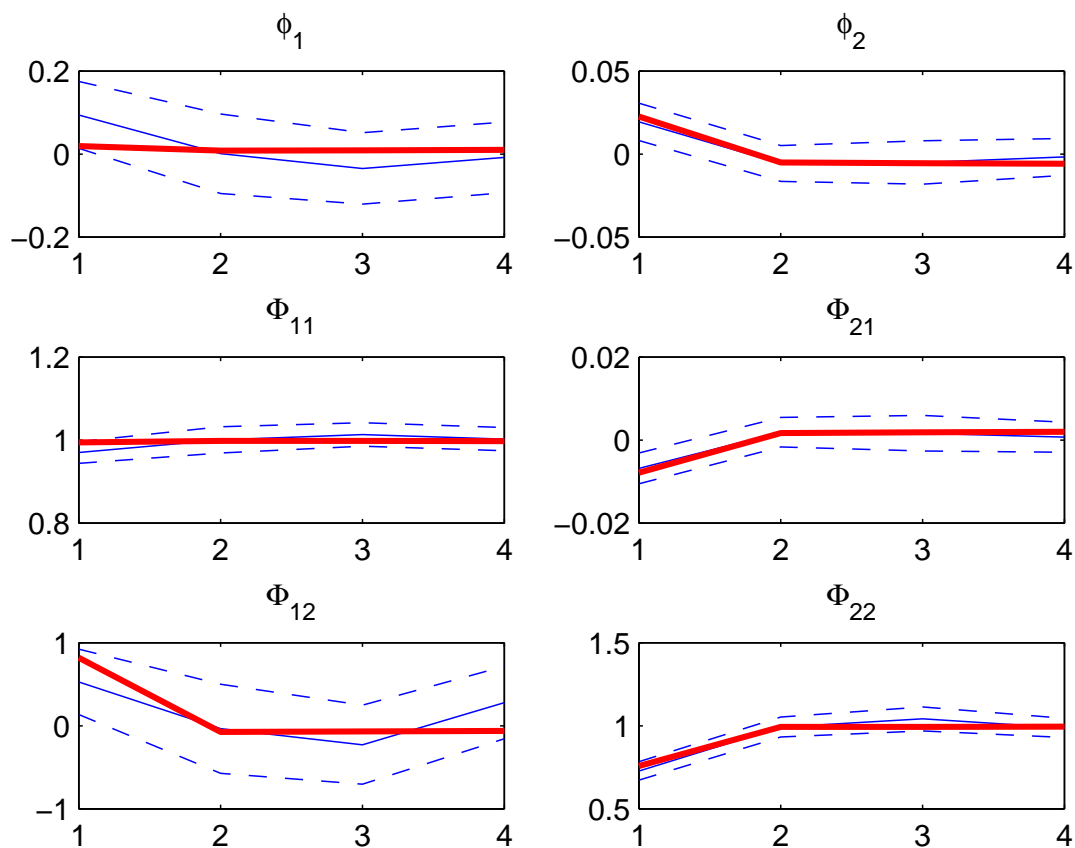


Figure 5:  $\pi_\Phi$  before 2005

Light blue line: Unrestricted OLS estimates of coefficients for regression in which  $y_{1t}$  is the dependent variable, plotted as a function of week of the month. Dashed blue lines: 95% confidence intervals for unrestricted OLS estimates. Bold red line: predicted values for coefficients derived from baseline model. All estimates based on data January 1990 to December 2004.

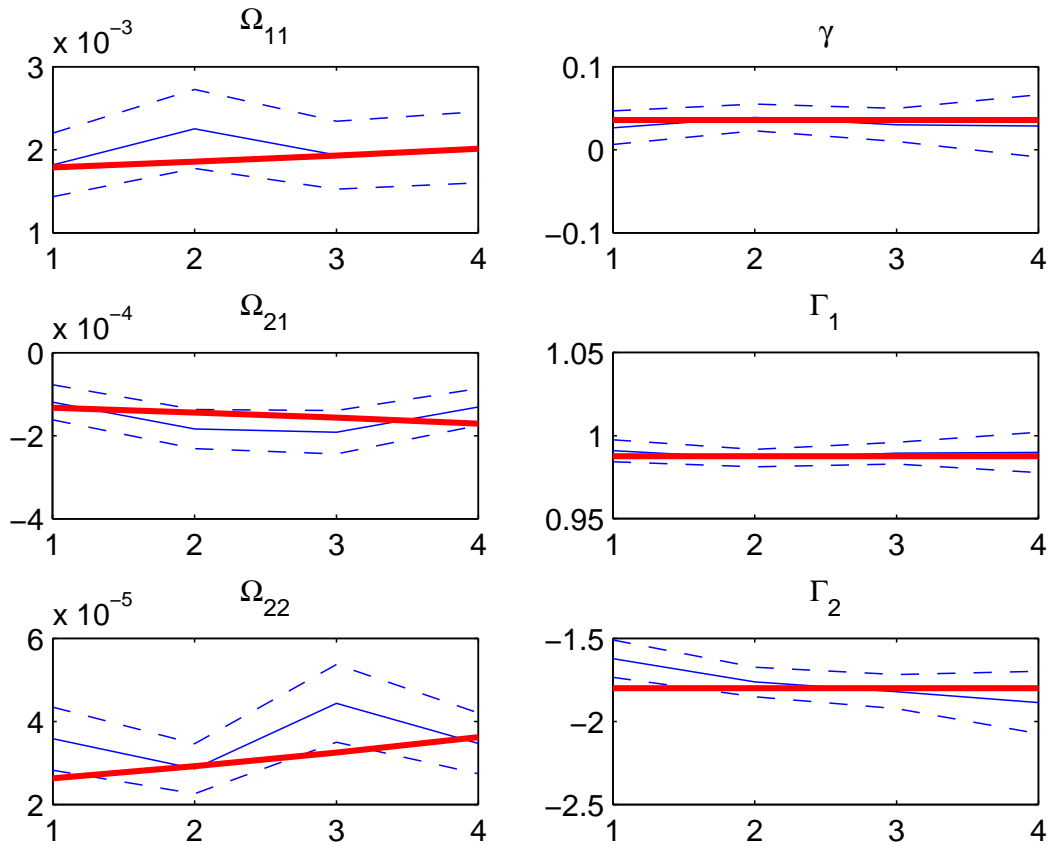


Figure 6:  $\pi_{\Omega}$  and  $\pi_{\Gamma}$  before 2005

First column: estimated elements of variance-covariance matrix for regression in which  $y_{1t}$  is the dependent variable, plotted as a function of week of the month. Second column: Estimated values of coefficients for regression in which  $y_{2t}$  is the dependent variable, plotted as a function of week of the month. In each panel, light blue lines are unrestricted OLS estimates, dashed blue lines are 95% confidence intervals for unrestricted OLS estimates, and bold red lines are predicted values for coefficients derived from baseline model. All estimates based on data January 1990 to December 2004.

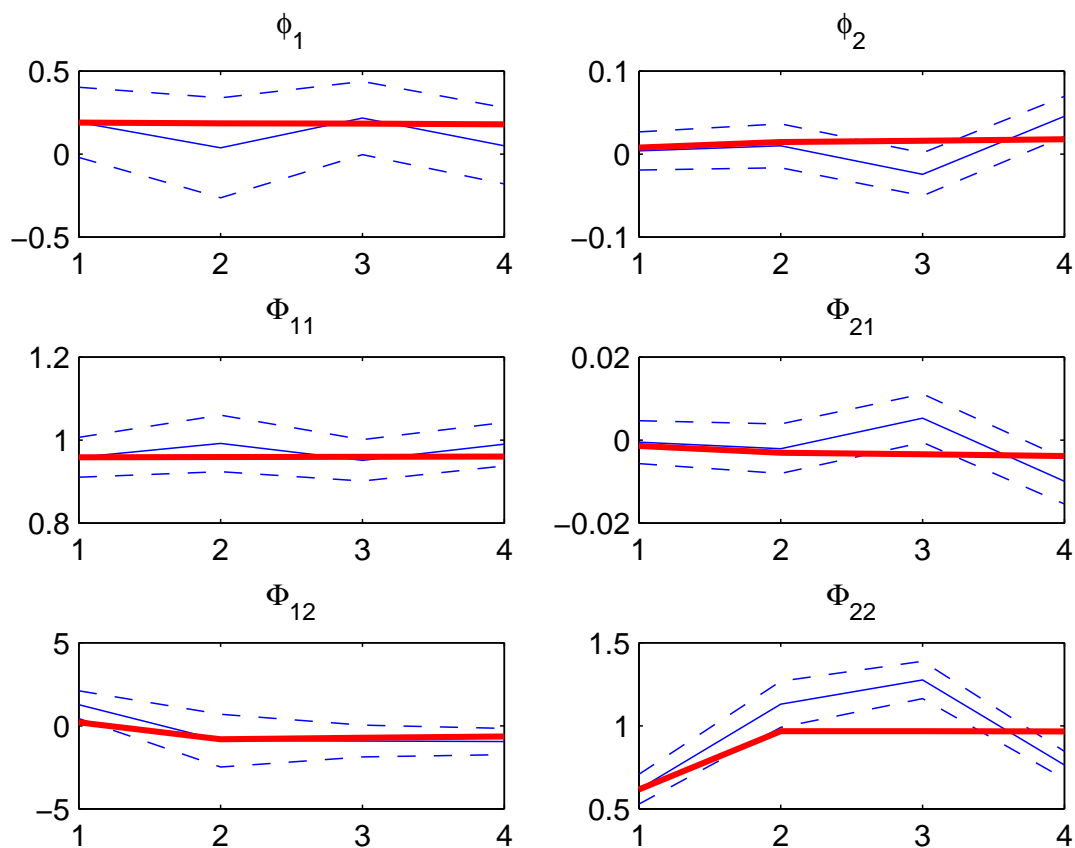


Figure 7:  $\pi_\Phi$  since 2005

Light blue line: Unrestricted OLS estimates of coefficients for regression in which  $y_{1t}$  is the dependent variable, plotted as a function of week of the month. Dashed blue lines: 95% confidence intervals for unrestricted OLS estimates. Bold red line: predicted values for coefficients derived from baseline model. All estimates based on data January 2005 to June 2011.

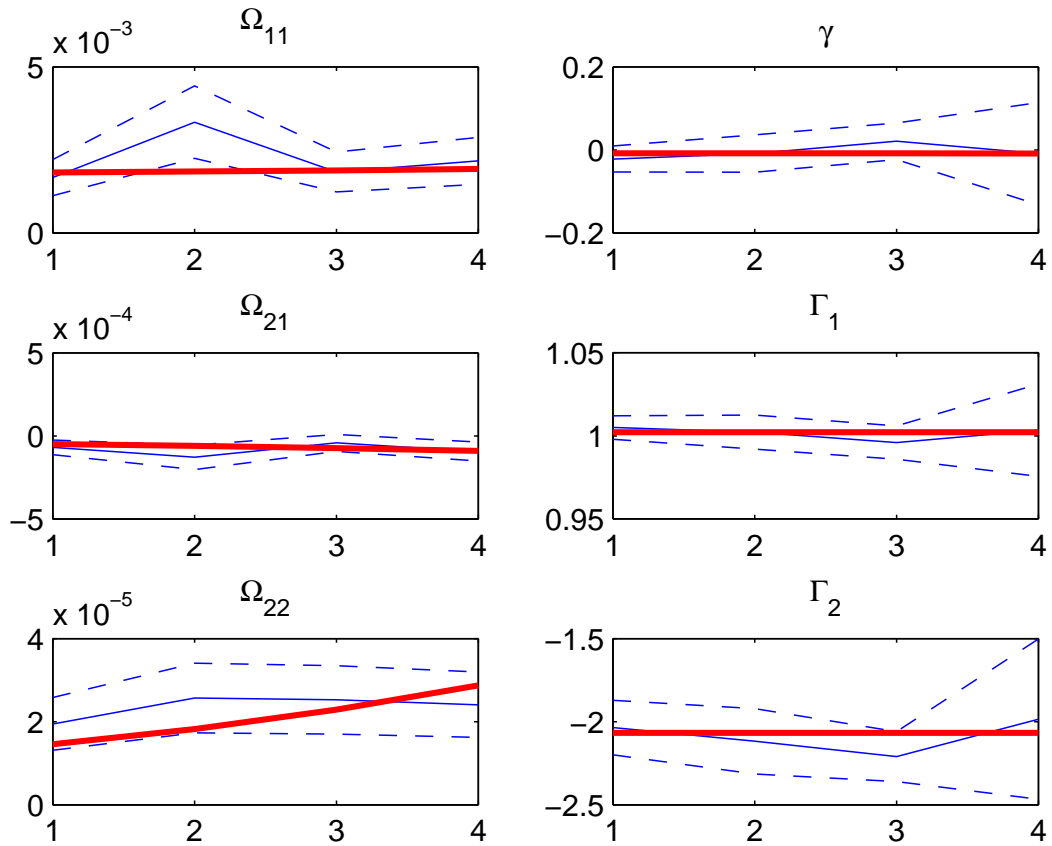


Figure 8:  $\pi_{\Omega}$  and  $\pi_{\Gamma}$  since 2005

First column: estimated elements of variance-covariance matrix for regression in which  $y_{1t}$  is the dependent variable, plotted as a function of week of the month. Second column: Estimated values of coefficients for regression in which  $y_{2t}$  is the dependent variable, plotted as a function of week of the month. In each panel, light blue lines are unrestricted OLS estimates, dashed blue lines are 95% confidence intervals for unrestricted OLS estimates, and bold red lines are predicted values for coefficients derived from baseline model. All estimates based on data January 2005 to June 2011.