Please answer all questions. Each of the four questions marked with a big number counts equally. Designate your answers clearly.

Recall the following definitions, concerning subsets of $\mathbf{R}^N$:

- a set is 'closed' if it contains all of its cluster points (limit points).
- a set is 'open' if, for each point in the set, there is an N-dimensional ϵ-ball, $\epsilon > 0$, (neighborhood) centered at the point, contained in the set.
- a set is 'bounded' if it can be contained in a cube of finite size, centered at the origin.
- a set is 'compact' if it is both closed and bounded.
- a set is 'convex' if for every two points in the set, the set includes the line segment connecting them.

1

1. Is the following subset of $\mathbf{R}^2$ closed? open? bounded? compact? convex? Explain your answer.

   $L = 30^o$ line through $(0, 2)$ $= \{(x, y)|(x, y) \in \mathbf{R}^2, y = \frac{1}{2}x + 2\}$.

   Suggested Answer: Closed: contains its limit points; Not open: contains no open balls in $\mathbf{R}^2$; unbounded: a line goes on forever; Not compact: closed but unbounded; Convex: contains the line segment between any two points of the line.

2. Is the following subset of $\mathbf{R}^2$ closed? open? bounded? compact? convex? Explain your answer.

   U = Union of two disjoint balls of radius 10, centered at the origin and at (20, 20), including their boundaries =

   $\{(x, y)|(x, y) \in \mathbf{R}^2, x^2+y^2 \leq 100\} \cup \{(x, y)|(x, y) \in \mathbf{R}^2, (x-20)^2+(y-20)^2 \leq 100\}$

   Suggested Answer: Closed: contains its limit points, the weak inequality defining the balls assures that; Not open, points on the boundary are not surrounded by open balls contained in U; Bounded: can be contained in a box of size 60×60 centered at the origin; Compact: both closed and bounded; Non-Convex: A line segment from the ball centered at the origin to the one centered at $(20, 20)$ passes out of the set U.

2

The Brouwer Fixed Point Theorem says that if S is a compact convex subset of $\mathbf{R}^N$ and if f is continuous, $f : S \to S$, then there is $x^* \in S$ so that $f(x^*) = x^*$; x^* is a fixed-point of the mapping f. For the following combinations of f and S, does the Brouwer theorem apply and does f have a fixed point? Explain your answer.

1. $S = \mathbf{R}^N_+$ (the nonnegative quadrant of $\mathbf{R}^N$), $w \in \mathbf{R}^N_+, w >> 0$ $(w_n > 0, n = 1, 2, \cdots, N)$, $f(x) = x + w$.

   Suggested Answer: The theorem does not apply and the the function does not have a fixed point. f is a continuous mapping from S into S but $S = \mathbf{R}^N_+$ is not compact. f maps each point into a translation of itself, with no point fixed in place.

2. $S =$ Hollow sphere of radius 10 centered at the origin $= \{(x, y)|(x, y) \in \mathbf{R}^2, x^2 + y^2 = 100\}$.

   $f(x, y) = -(x, y)$. f maps each point of the sphere to its diametric opposite point.

   Suggested Answer: The theorem does not apply and the function has no fixed point. f is a continuous mapping from S into S but $S =$ a hollow sphere is not convex. f maps each point into its opposite, with no point fixed in place.

3

In a general equilibrium model, when prices adjust so that markets clear (the general equilibrium prices are achieved) and the resulting allocation is Pareto efficient, the price system is said to "decentralize the efficient allocation." What does "decentralize" mean in this statement?

Suggested Answer: Typically in a general equilibrium model, there is strong interdependence among households and between households and firms. Households consume what firms produce. One household's consumption of a good in limited supply depends on others' not taking it for themselves. Nevertheless, in a general equilibrium, firms and households do not take others' actions into account. They merely consult prevailing prices. This lack of specific co-ordination among firms, among households, and between firms and households is described as 'decentralization.' Though each firm's and household's opportunities depend on others' actions, they need not take them into account. They rely merely on prices to provide sufficient co-ordination.

4

Show that the following sequences in $\mathbf{R}$ are not convergent, but find a convergent subsequence.

1. $x^\nu = 3^\nu + (-1)^\nu 3^\nu, \nu = 1, 2, 3, \ldots$ That is, for ν odd, $x^\nu = 0$, for ν even, $x^\nu = 2 \times (3^\nu)$ where 3^ν is 3 raised to the power ν.

   Suggested Answer: The sequence is bouncing between 0 and positive values becoming arbitrarily large. It is not converging to anything. But the odd-numbered subsequence is a constant. It is trivially convergent to the constant value of 0. Restating, $x^\nu \to 0$, for $\nu = 1, 3, 5, 7, \cdots$.

2. $x^\nu \in \mathbf{R}$, $\nu = 1, 2, 3, \ldots$. $x^\nu = (-1)^\nu(10)$. That is, $x^\nu = -10$ for ν odd and $x^\nu = 10$ for ν even.

   Suggested Answer: The sequence is oscillating between 10 and -10. There is no single limiting value. But each of the even- and odd-numbered subsequences is constant and hence convergent to its constant value. The even-numbered subsequence is converging to 10. The odd-numbered subsequence is converging to -10. Restating this, $x^\nu \to 10$ for $\nu = 2, 4, 6, 8, \cdots$; $x^\nu \to -10$ for $\nu = 1, 3, 5, 7, \cdots$.