# OPTIMAL PRODUCTION AND ALLOCATION UNDER UNCERTAINTY *

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## I. Ex Ante and Ex Post Optimum

Every equilibrium in a perfectly competitive economy where all agents have complete information is Pareto optimal.[1] Difficulties that might arise in reaching this conclusion because of uncertainty about the future were eliminated by a clever device due to Arrow.[2] If one is uncertain about the future, at least one can make an exhaustive list of conceivable future states of the world.[3] Indentical commodities existing in different states of the world are distinct goods, have different prices, and enter separately in individuals' preferences and firms' productions.[4] Trade is not in goods but in contracts for delivery of goods contingent on whether a given state of the world occurs. The relevant characteristics of the analytic structure developed for the case of certainty remain unchanged. Indeed:

> The formal identity of this theory of uncertainty with the theory of certainty developed earlier allows one to apply all the results established. . . . In particular, sufficient conditions for the existence of an equilibrium for the private ownership economy . . . are [the same].

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1. G. Debreu, *Theory of Value* (New York: Wiley, 1959).
2. K. J. Arrow, "Le Role des Valeurs Boursières pour la Repartition la Meilleure des Risques," *International Colloquium on Econometrics, 1952,* Centre National de la Recherche Scientifique, Paris, 1953. Translated as "The Role of Securities in the Optimal Allocation of Risk-Bearing," *Review of Economic Studies*, vol. 31 (April 1964).
3. See L. J. Savage, *Foundations of Statistics* (New York: Wiley, 1954).
4. Debreu, *op. cit.*, Ch. 7.

> In the same fashion, [previously established] theorems . . . yield sufficient conditions for an equilibrium relative to price system to be an optimum, and for an optimum to be an equilibrium relative to a price system.[5]

Formally this is true, but there are differences so fundamental between the certainty and uncertainty economies that the actual significance of the results belies their superficial similarity.

Suppose that consumer choices are made so as to maximize *expected* utility under budget constraint. The configuration achieved is such that no trader's expected utility can be increased by a redistribution of contingent commodities (contracts deliverable at a certain date and event) without decreasing some trader's expected utility. Such a situation is known as an Arrow optimum, constituting an "optimal allocation of risk bearing."[6] Professor Radner aptly describes the situation as "optimum relative to a given structure of information in the economy."[7]

As a practical matter, the achievement of Arrow optimum is a normative dead end. After all, we are not so much interested in expectations as in results. Given an Arrow optimal distribution of contingent claims and supposing the occurrence of some event, we can then ask whether in that event the distribution of real goods resulting from the given distribution of contingent claims is a Pareto optimal distribution of real goods. If the answer is "no," then it is comparatively small comfort to know that the economy had achieved an optimal allocation of risk bearing. If we are interested in satisfactions actually realized rather than those that are merely anticipated, the appropriate quality to seek is that there be no redistribution that will increase some trader's realized utility while decreasing no trader's realized utility. Such a situation will be termed an ex post Pareto optimum. Depending on the structure of subjective probabilities and on the events that occur, there may be Arrow optima that are not ex post Pareto optima and ex post Pareto optima that are not Arrow optima. A situation is said to be an intratemporal Pareto optimum if there is no feasible redistribution or reallocation of goods all of a single time period and event,

5. *Ibid.*, p. 102.

6. Priority for enunciation of the distinction between ex ante and ex post optimum and for independent discovery of some of the results of section IV in the case of a social welfare function whose arguments are differentiable utility functions is due to Jacques Drèze in "Market Allocation Under Uncertainty" (paper presented at the First World Congress of the Econometric Society, Rome, Sept. 1965; abstract in *Econometrica*, vol. 34, no. 5 (supplementary issue 1966), p. 42). This usage follows from the optimality properties studied by Arrow, *op. cit.*

7. R. Radner, "Competitive Equilibrium under Uncertainty," *Econometrica*, vol. 36 (Jan. 1968), pp. 31–58.

outputs of other periods and events remaining fixed, such that some trader is made better off and no trader is made worse off.

## II. Consumers and Commodities

Most of the interesting questions on uncertainty and optimality over time can be meaningfully posed and answered in a two-period model. The effect of considering more periods would be primarily to introduce more complicated and confusing notation. In the first period let there be a unique state of the world known to all traders. Let the conceivable states of the world in the second period comprise the elements of the finite set $S$.[8] Let there be $n$ goods available in each state and period. Then there are $(|S|+1)n$ commodities (period 1 goods and period 2 contingent commodities) traded ex ante.[9]

A commodity bundle then is an element of $\Omega_A$, the nonegative orthant of Euclidean $(|S|+1)n$ space. An ex ante price vector is also an element of $E^{(|S|+1)n}$. Eventually realized bundles will be elements of $\Omega_P$, the nonnegative orthant of Euclidean $2n$ space (as is an ex post price vector). Traders are elements of the finite set $T$. For each $t \epsilon T$ there is a utility function,[1] $u^{tj}(x^{t1}, x^{2tj})$, defined for each $j \epsilon S$ and for all arguments within some bounded (feasible) subset of the nonnegative orthant of $E^{2n}$. $t$'s utility function is allowed to vary with the state of the world prevailing in period 2. This reflects the possibility that the satisfaction derived from an umbrella may depend on the weather. I will suppose that $u^{tj}$ satisfies Arrow's assumptions 2, 3, and 6.[2] These include free disposability and strict concavity.[3] This combination has the advantage of assur-

8. There are some subtleties in the specification and even the existence of $S$ investigated in Radner, *op. cit.*, and J. Hirshleifer, "The Private and Social Value of Information and the Reward to Inventive Activity," Working Paper No. 158, Western Management Science Institute, University of California, Los Angeles, April 1970, and *American Economic Review*, vol. 61 (Sept. 1971), pp. 561–75.

9. $|S|$ denotes the number of elements in $S$. I will take $S$ to be finite (but see T. Bewley, "Equilibrium Theory with an Infinite Dimensional Commodity Space," Center for Research in Management Science, University of California, Berkeley, 1970).

1. In order meaningfully to talk of maximizing expected utility, one must assume cardinal properties of the utility function. Since one degenerates into nontheory without so disreputable an assumption, there is little choice.

2. K. J. Arrow, "An Extension of the Basic Theorems of Classical Welfare Economics," in *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability,* J. Neyman, ed. (Berkeley: University of California Press, 1951).

3. Concavity in this context implies risk aversion. For a discussion of this point see Arrow, "The Role of Securities," *op. cit.* There is no significant loss of generality in requiring strict concavity rather than concavity.

ing strict positivity of price vectors, which makes them easier to work with than might otherwise be the case. A further simplification is introduced by assuming utilities to be separable over time. Thus, assume

(1) $u^{tj}(x^{t1}, x^{t2j}) = v_1{}^t(x^{t1}) + v_2{}^{tj}(x^{t2j})$,

for all $t \epsilon T$, $j \epsilon S$.

Each trader has a subjective probability of occurrence of state $j$, for each $j \epsilon S$. Denote this quantity by $f(t, j)$. Then the expected utility of an ex ante bundle $x^t$ for trader $t$ is

(2) $$E^t u^t(x^t) = \sum_{j \epsilon S} f(t, j) u^{tj}(x^{t1}, x^{t2j}).$$

By (1)

(3) $$E^t u^t(x^t) = v_1{}^t(x^{t1}) + \sum_{j \epsilon S} f(t, j) v_2{}^{tj}(x^{t2j}).$$

Thus a distribution $x^t$, $t \epsilon T$ is said to be an Arrow optimum if there is no feasible $y^t$, $t \epsilon T$ so that

(4) $E^t u^t(x^t) \leqslant E^t u^t(y^t)$ for all $t \epsilon T$

with the strict inequality holding for at least one $t \epsilon T$.

Ex ante trader $t \epsilon T$ will prefer $y^t$ to $x^t$ if and only if

(5) $E^t u^t(x^t) < E^t u^t(y^t)$;

equivalently

(6) $$v_1{}^t(x^{t1}) + \sum_{j \epsilon S} f(t, j) v_2{}^{tj}(x^{t2j}) < v_1{}^t(y^{t1}) + \sum_{j \epsilon S} f(t, j) v_2{}^t(y^{t2j});$$

equivalently

(7) $$\sum_{j \epsilon S} f(t, j) [v_2{}^{tj}(x^{t2j}) - v_2{}^{tj}(y^{t2j})] < v_1{}^t(y^{t1}) - v_1{}^t(x^{t1}).$$

If the $y^t$ differs from $x^t$ only in first-period consumption and consumption in period 2 for event $j^* \epsilon S$, ex ante $y^t$ will be preferred to $x^t$, if and only if

(8) $f(t, j^*) [v_2{}^{tj^*}(x^{t2j^*}) - v_2{}^{tj^*}(y^{t2j^*})] < v_1{}^t(y^{t1}) - v_1{}^t(x^{t1})$.

Similarly, ex post, if state $j^*$ obtains, $t$ prefers $y^t$ to $x^t$ if and only if

(9) $v_2{}^{tj^*}(x^{t2j^*}) - v_2{}^{tj^*}(y^{t2j^*}) < v_1{}^t(y^{t1}) - v_1{}^t(x^{t1})$.

Thus a distribution $x^t$, $t \epsilon T$, is said to be ex post Pareto optimal for state $j^*$ if there is no feasible distribution $y^t$, $t \epsilon T$, so that

(10) $v_2{}^{tj^*}(x^{t2j^*}) - v_2{}^{tj^*}(y^{t2j^*}) \leqslant v_1{}^t(y^{t1}) - v_1{}^t(x^{t1})$

for all $t \epsilon T$ with the strict inequality holding for at least one $t \epsilon T$.

Finally, $x^t$, $t \epsilon T$, is intratemporal Pareto optimal for state $j^*$ if there exists neither $y^t$ nor $z^t$ feasible, $t \epsilon T$, so that $y^{t1} = x^{t1}$, $y^{t2j} = x^{t2j}$ for all $t \epsilon T$, all $j \epsilon S - \{j^*\}$, $z^{t2j} = x^{t2j}$ for all $t \epsilon T$, all $j \epsilon S$, and

(11) $v_2{}^{tj^*}(x^{t2j^*}) - v_2{}^{tj^*}(y^{t2j^*}) \leqslant 0$

or

(12) $v_1{}^t(z^{t1}) - v_1{}^t(x^{t1}) \geqslant 0$

with the strict inequality holding for at least one $t \epsilon T$.

## III. Consumer Choice and the Subjectivity Theorem

Given an ex ante price system $p \epsilon \Omega_A$, the problem of choice for $t \epsilon T$ is merely to choose $x^t \epsilon \Omega_A$ so that $x^t$ maximizes $t$'s expected utility subject to budget constraint. If $t$'s choice is $x^{\theta t}$ we know that

(13) $E^t u^t(x^{\theta t}) \geqslant E^t u^t(x^t)$ for all $x^t \epsilon \Omega_A$

such that $p \cdot x^t \leqslant p \cdot x^{\theta t}$. In particular if $p \cdot y \leqslant p \cdot x^{\theta t}$ and $y$ differs from $x^{\theta t}$ only in its period 1 and 2, state $j$ coordinates; then $E^t u^t(x^{\theta t}) \geqslant E^t u^t(y)$ if and only if

(14) $f(t, j)[v_2{}^{tj}(y^{2j}) - v_2{}^{tj}(x_\theta{}^{t2j})] \leqslant v_1{}^t(x_\theta{}^{t1}) - v_1{}^t(y^1).$

$x^{\theta t}$ may be a maximizing choice under prices $p$ even for other subjective probabilities.[4] It is convenient to know the range of such subjective probabilities and how they depend on the supporting prices. Thus, let

$$\begin{aligned} K(t, x^t, j) = \{ (p, k) \ : \ & p \epsilon \Omega_P, p >> 0, 0 < k \leqq 1, \\ & k[v^{tj}(z^2) - v_2{}^{tj}(x^{t2j})] \leqq v_1{}^t(x^{t1}) - v_1{}^t(z^1) \\ & \text{for all } (z^1, z^2) \epsilon \Omega_P \text{ such that} \\ & (p^1, p^2) \cdot (z^1, z^2) \leqq (p^1, p^2) \cdot (x^{t1}, x^{t2j}) \}. \end{aligned}$$

Definition: *Let* $\mathrm{x^t}$, $\mathrm{t \epsilon T}$, *be an Arrow optimum. Subjective probabilities for state* j *are said to be universally similar at* $\mathrm{x^t}$, $\mathrm{t \epsilon T}$, *if* $\bigcap_{\mathrm{t \epsilon T}} \mathrm{K(t, x^t, j)}$ *is nonempty.*

That is, subjective probabilities for state $j$ are universally similar if there is a value $k$ and a price system $p$ such that if all traders agreed that their subjective probabilities of $j$'s occurrence equaled $k$, then under prices $p$ they would have no reason to change their state $j$ consumption choices from $x^t$. Subjective probabilities are similar if there is a value of the subjective probability of state $j$ that, if commonly held, would be consistent with the allocation $x^t$, $t \epsilon T$, and a price system. Subjective probabilities for state $j$ are universally similar whenever they are equal for all $t \epsilon T$.

Theorem 1, Subjectivity of Arrow Optimum: *Let* $\mathrm{x^t}$, $\mathrm{t \epsilon T}$, *be an Arrow optimum. Suppose subjective probabilities for state* j

4. In the special case of differentiable utility functions this problem would not arise. However, the simplification that would be introduced by assuming differentiability does not seem worth the loss of generality. See Section IV below.

*are not universally similar at* $x^t$, $t \epsilon T$. *Let* $0 < k \leqslant 1$. *Then there is* $z^t \epsilon \Omega_A$, *for each* $t \epsilon T$, $\sum_{t \epsilon T} z^t = \sum_{t \epsilon T} x^t$, *so that*

$$k[v_2^{tj}(z^{t2j}) - v_2^{tj}(x^{t2j})] \geqq v_1^t(x^{t1}) - v_1(z^{t1}), \textit{ for all } t \epsilon T,$$

*with the strict inequality holding for at least one* $t \epsilon T$.

*Proof*: Consider the pure exchange economy consisting of the set of traders $T$ with utility functions $\mu^t(w^t) = v_1^t(w^{t1}) + kv_2^{tj}(w^{t2})$ for $w^t \epsilon \Omega_P$. Let the economy's total endowment be $\sum_{t \epsilon T} (x^{t1}, x^{t2j})$. Then the distribution $(x^{t1}, x^{t2j})$, $t \epsilon T$, is not a Pareto optimum for this economy. This follows since the absence of universal similarity implies that there is no supporting price vector which is a necessary condition for Pareto optimum.[5] That is, there is no

$$(p^1, p^2) \epsilon \Omega_P$$

so that

$$v_1^t(x^{t1}) + kv_2^{tj}(x^{t2j}) \geqq v_1^t(z^{\theta t1}) + kv_2^{tj}(z^{\theta t2})$$

for all

$$(z^{\theta 1}, z^{\theta 2}) \epsilon \Omega_P$$

so that

$$(p^1, p^2) \cdot (z^{\theta t1}, z^{\theta t2}) \leqq (p^1, p^2) \cdot (x^{t1}, x^{t2j}).$$

Thus, there is

$$(z^{\theta t1}, z^{\theta t2}), \ t \epsilon T,$$

such that

$$\sum_{t \epsilon T} (z^{\theta t1}, z^{\theta t2}) = \sum_{t \epsilon T} (x^{t1}, x^{t2j})$$

so that

$$v_1^t(x^{t1}) + kv_2^{tj}(x^{t2j}) \leqq v_1^t(z^{\theta t1}) + kv_2^{tj}(z^{\theta t2})$$

for all $t \epsilon T$ with the strict inequality for at least one $t \epsilon T$. Let

$$z^{t1} = z^{\theta t1}, \ z^{t2j} = z^{\theta t2}, \ z^{t2i} = x^{t2i}$$

for $i \epsilon S - \{j\}$. Then $z^t \epsilon \Omega_A$, for all $t \epsilon T$,

$$\sum_{t \epsilon T} x^t = \sum_{t \epsilon T} z^t,$$

and

$$k[v_2^{tj}(z^{t2j}) - v_2^{tj}(x^{t2j})] \geqq v_1^t(x^{t1}) - v_1^t(z^{t1})$$

for all $t \epsilon T$, with the strict inequality holding for at least one $t \epsilon T$.

Q.E.D.

The subjectivity theorem asserts that information affects action. If subjective probabilities are not similiar, then information changing them all (to $k$) will result in changed consumption decisions as well.

5. By Arrow, "An Extension of the Basic Theorems," *op. cit.*

THEOREM 2: *Any Arrow optimum with* f(t, j) > 0 *for at least one* t$\epsilon$T *for each* j$\epsilon$S *is an intratemporal Pareto optimum.*

*Proof*: Without loss of generality consider only period 2, arbitrary $j^* \epsilon S$. By Arrow optimality there is no $z^t$, $t \epsilon T$, such that

$$\sum_{t \epsilon T} z^t \epsilon Y, \; E^t u(z^t) \geqslant E^t u(x^{\theta t})$$

for all $t \epsilon T$ with the strict inequality holding for at least one $t$ in $T$. In particular this means there is no such $z^t$ differing from $x^{\theta t}$ only in period 2 state $j^*$. That is, there is no $z^t$ feasible so that

$$(15) \qquad z^{t1} = x^{\theta t1};$$

$$(16) \qquad z^{t2j} = x^{t2j} \text{ for all } t \epsilon T, \text{ all } j \epsilon S - \{j^*\};$$

so that

$$(17) \qquad f(t, j^*) \; v_2^{tj^*}(z^{t2j^*}) \geqslant f(t, j^*) v_2^{tj^*}(x^{\theta t2j^*}) \text{ for all } t \epsilon T$$

with the strict inequality holding for at least one $t \epsilon T$.

For those $t \epsilon T$ such that $f(t, j^*) > 0$, (17) implies that there is no $z^t$ feasible fulfilling (15) and (16) for all $t \epsilon T$, all $j \epsilon S - \{j^*\}$ so that

$$(18) \qquad v_2^{tj^*}(z^{t2j^*}) \geqslant v_2^{tj^*}(x^{\theta t2j^*})$$

with the strict inequality holding for some such $t \epsilon T$. This combined with strict nonsatiation [6] implies that there is no $z^t$ feasible fulfilling (15) and (16) as above such that

$$(19) \qquad v_2^{tj^*}(z^{t2j^*}) = v_2^{tj^*}(x^{\theta t2j^*})$$

for all $t$ with $f(t, j^*) > 0$ and $z^t \geqslant 0$ and $z^t \neq 0$ for some $t$ such that $f(t, j^*) = 0$. If the latter were the case, then (18) could be fulfilled with strict inequality for some $t$.

Thus,

$$v_2^{tj^*}(x^{\theta t2j^*}) \geqslant v_2^{tj^*}(z^{t2j^*})$$

for all feasible $z^t$, satisfying (15), (16) for all $t \epsilon T$, $j \epsilon S - \{j^*\}$.

Q.E.D.

Theorem 2 says that, given Arrow optimum, any lack of ex post Pareto optimality that arises under uncertainty over time is not due to nonoptimal distribution of chosen output within a given time and event, but rather comes from misallocation and maldistribution over time or across events. Thus, an economy that has Arrow optimal distribution will appear to have Pareto optimal production and distribution at any point in time. It is only with respect to the choice of present versus future consumption that nonoptimality arises.

6. Arrow, "An Extension of the Basic Theorems," *op. cit.*, Lemma 10.

## IV. Necessary Conditions for Ex Post Pareto Optimum Given Arrow Optimum

Given Arrow optimum, a necessary condition for ex post Pareto optimum is that subjective probabilities for the state that actually occurs should be universally similar.[7]

Theorem 3: *Let* $x^t$, $t \epsilon T$, *be an Arrow optimum. Let* $j^*$ *be the event that occurs,* $f(r, j^*) > 0$ *for some* $r \epsilon T$. *A necessary condition for* $(x^{t1}, x^{t2j^*})$, $t \epsilon T$, *to be an ex post Pareto optimum is that subjective probabilities for* $j^*$ *be universally similar.*

*Proof*: Suppose universal similarity does not hold. By Theorem 1 there is a feasible distribution $z^t$, $\sum_{t \epsilon T} z^t = \sum_{t \epsilon T} x^t$, such that

$$(20) \qquad k[v_2^{tj*}(z^{t2j*}) - v_2^{t2j*}(x^{t2j*})] \geqslant v_1^t(x^{t1}) - v_1^t(z^{t1}) \text{ for all } t \epsilon T$$

with a strict inequality for at least one $t \epsilon T$. Concavity of $v_2^{tj*}$ gives

$$(21) \qquad v_2^{tj*}[(1-k)x^{t2j*} + kz^{t2j*}] \geqslant (1-k)v_2^{tj*}(x^{t2i*}) + kv_2^{tj*}(z^{t2j*}) = k[v_2^{tj*}(z^{t2j*}) - v_2^{tj*}(x^{t2j})] + v_2^{tj*}(x^{t2j*}).$$

With (20), this gives

$$(22) \qquad v_2^{tj*}[(1-k)x^{t2j*} + kz^{t2j*}] - v_2^{tj*}(x^{t2j*}) \geqslant v_1^t(x^{t1}) - v_1^t(z^{t1}).$$

A strict inequality holds in (22) for some $t \epsilon T$. But

$$(z^{t1}, (1-k)x^{t2j*} + kz^{t2j*}), \ t \epsilon T,$$

is a feasible distribution and by (22) it is ex post Pareto preferable to $(x^{t1}, x^{t2j*})$. Therefore $(x^{t1}, x^{t2j*})$ is not an ex post Pareto optimum. Q.E.D.

Corollary 3.1: *Let* $x^t$, $t \epsilon T$, *be an Arrow optimum. Suppose for all* $j \epsilon S$ *subjective probabilities for state* $j$ *are not universally similar. Then for any* $j \epsilon S$, $(x^{t1}, x^{t2j})$ $t \epsilon T$ *is not an ex post Pareto optimum.*

The above theorem is fundamental to understanding how information and subjective probabilities affect optimal allocation under uncertainty. If subjective probabilities for the state that occurs differ significantly (i.e., so that universal similarity does not hold), then ex post misallocation will definitely result. The reason is that ex ante diversity of subjective probabilities implies ex post diversity of marginal rates of substitution of present versus future consumption; equality of these rates is a necessary condition for Pareto optimum.

7. See also Drèze, *op. cit.*

The corollary points out that if subjective probabilities lack universal similarity for all states of the world, then no matter what state of the world occurs, the resulting distribution will not be an ex post Pareto optimum. This is not to say that there is a redistribution ex ante that will make all traders better off ex post no matter what state of the world takes place. If this were the case, the original allocation would not have been an Arrow optimum since redistribution would have increased all traders' expected utility. Rather, the corollary says that, given sufficient diversity, it is clear that ex post Pareto nonoptimality will result.

If utility functions are differentiable the mathematics of the problem is particularly straightforward. One can rely on the theorem that whenever goods are consumed in nonzero quantities by two traders in a market, a necessary condition for Pareto optimum is that marginal rates of substitution for the two goods be the same for the two traders. Consider a pure exchange economy with one good denoted $c$. Let the two traders in question be $t, r$. Then Arrow optimality implies that $t$'s marginal rate of substitution ex ante of a contingent claim due period 2 state $j$ for certain good period 1 must equal $r$'s marginal rate of substitution. Thus,

$$f(t, j)\frac{\dfrac{dv_2^{tj}(c_2^{\theta tj})}{dc_2^{tj}}}{\dfrac{dv_1^{t}(c_1^{\theta t})}{dc_1^{t}}}=f(r, j)\frac{\dfrac{dv_2^{rj}(c_2^{\theta rj})}{dc_2^{rj}}}{\dfrac{dv_1^{r}(c_1^{\theta r})}{dc_1^{r}}}$$

for all $j\epsilon S$, where $c^{\theta t}$ is $t$'s consumption choice. When state $j^*$ occurs, ex post Pareto optimality requires

$$\frac{\dfrac{dv_2^{tj}(c_2^{\theta tj*})}{dc_2^{tj*}}}{\dfrac{dv_1^{t}(c_1^{\theta t})}{dc_1^{t}}}=\frac{\dfrac{dv_2^{rj*}(c_2^{\theta rj*})}{dc_2^{rj*}}}{\dfrac{dv^{r}(c_1^{\theta r})}{dc_1^{r}}}.$$

Clearly, this will obtain if and only if $f(t, j^*)=f(r, j^*)$ for all $t, r\epsilon T$.

## V. Production

Production is characterized by a set of feasible output bundles. Let $Y$ be the ex ante transformation set. $Y$ is a convex subset of $\Omega_A$. An element $y\epsilon Y$ denotes a combination of feasible period 1 outputs and feasible period 2 state $j$ outputs for each $j\epsilon S$. A distribution $x^t \epsilon \Omega_A$ for each $t \epsilon T$ is said to be feasible if $\sum_{t\epsilon T} x^t \epsilon Y$. $Y_j$ is a convex subset of $\Omega_P$ for each $j \epsilon S$. $y \epsilon Y_j$ denotes a feasible combination of period 1 and period 2 outputs in the case that $j \epsilon S$ is the state that occurs in period 2.

Assume that if $(y^1, y^2) \epsilon Y_j$ there is $z \epsilon Y$ so that $z^1 = y^1$, $z^{2j} = y^2$. This says nothing more than that no output feasible ex post for given $j \epsilon S$ is infeasible ex ante. I will also assume [8] that for any $y \epsilon Y$ there is $y_j \epsilon Y_j$ so that $y_j \geqslant (y^1, y^{2j})$. That is, one assumes that consideration of several possible events does not increase possible output in any event over what it would be in the case of uncertainty.

An output is said to be efficient if there is no other feasible output that is greater in some component and less in no component.

DEFINITION: $\mathrm{y} \epsilon \mathrm{Y}$ *is efficient if and only if there is no* $\mathrm{y}^\theta \epsilon \mathrm{Y}$ *so that* $\mathrm{y}^\theta \geqslant \mathrm{y}$ *and* $\mathrm{y}^\theta \neq \mathrm{y}$. $\mathrm{y_j} \epsilon \mathrm{Y_j}$, $\mathrm{j} \epsilon \mathrm{S}$, *is efficient if there is no* $\mathrm{y}^\theta{}_\mathrm{j} \epsilon \mathrm{Y_j}$ *such* that $y_j{}^\theta \geqslant y_j$ and $y_j{}^\theta \neq y_j$.

DEFINITION: *Let* $\mathrm{y} \epsilon \mathrm{Y}$, $\mathrm{j} \epsilon \mathrm{S}$. *Then y is said to be* j*-efficient if* $(\mathrm{y}^1, \mathrm{y}^2\mathrm{j})$ *is efficient in* $\mathrm{Y_j}$.
*Let* $\mathrm{W_j} =$ *class of efficient points in* $\mathrm{Y_j}$.

DEFINITION: *Let* $\mathrm{w} \epsilon \mathrm{W_j}$, $\mathrm{y} \epsilon \mathrm{Y}$. *Consider the conditions,*

(a) $w^1 = y^1$,
  (a.i) $z \epsilon Y$,
  (a.ii) $z^1 = w^1 = y^1$, $z^{2i} = y^{2i}$, all $i \epsilon S - \{j\}$
(b) $w^2 = y^{2j}$
  (b.i) $z \epsilon Y$
  (b.ii) $z^{2j} = w^2 = y^{2j}$; $z^{2i} = y^{2i}$, all $i \epsilon S - \{j\}$,
(c) $(z^1, z^{2j}) \epsilon W_j$.
(d) $(w^1, w^2) = (z^1, z^2)$.

$Y$ is said to fulfill weak independence with respect to state $j$ if whenever (a) is fulfilled, there is $z$ fulfilling (a.i), (a.ii), (c); and whenever (b) is fulfilled, there is $z$ fulfilling (b.i), (b.ii), (c). $Y$ is said to fulfill strong independence with respect to state $j$ if $Y$ fulfills weak independence and condition (d).

Thus the production set has an independence property with respect to state $j$ if a $j$-efficient output can be achieved without re-arranging output for other states. Independence with respect to state $j$ holds if there is not much trade-off across events, when

8. $\geqslant$ denotes greater than or equal coordinatewise; $>>$ denotes strictly greater than coordinatewise; 0 denotes the zero vector or scalar depending on context.

feasibility of output for state $j$ is not much affected by output choice for other states of the world.

$Y$ is said to be intratemporal if

$$Y = Y^1 \times Y_1^2 \times Y_2^2 \times \ . \ . \ . \ \times Y^2_{|S|},$$

where $\times$ denotes Cartesian product; $Y^1$ is the period 1 transformation set; $Y_j^2$ is the period 2, state $j$ transformation set $j=1, \ . \ . \ . \ , |S|$. $Y$ is intratemporal if there is no trade-off in production possibilities among either events or time periods; admissible trade-offs are within a given time and event. A pure exchange economy is intratemporal.

DEFINITION: *Let* $x^t$, $t \in T$, *be an Arrow optimum. Let* $\sum_{t \in T} (x^{t1}, x^{t2j})$ *be efficient in* $Y_j(\sum_{t \in T} (x^{t1}, x^{t2j}) \in W_j)$. *Subjective probabilities for state* j *are said to be effectively similar if there is* $(k, p) \in \bigcap_{t \in T} K(t, x^t, j)$ *so that* $p \cdot (\sum_{t \in T} (x^{t1}, x^{t2j})) \geqq p \cdot y$ *for all* $y \in Y_j$.

That is, subjective probabilities are effectively similar if they are universally similar, and if some price consistent with universal similarity is also consistent with profit-maximizing production.

Subjective probabilities are effectively similar whenever they are equal for $x^t$ such that $\sum_{t \in T} (x^{t1}, x^{t2j}) \in W_j$.

## VI. EFFICIENCY

In most economies uncertainty has severe implications for the ex post optimality of production decisions. Drought-resistant seeds planted in a year that turns out to have a heavy rainfall will yield a disappointing harvest. Nothing short of good luck or good prediction will alleviate this problem. However, there is a moderately well-defined class of production sets under uncertainty in which producers are not forced to choose between maximal output in one state of the world versus maximal output in the other. One can have both. If production sets are of this form, then there is no particular value to good prediction. Producers can do just as well in ignorance. Conversely, in economies with production sets that are not in this class, there is likely to be substantial value to good prediction.

Productive efficiency is a necessary condition for optimum. An Arrow optimum is efficient in $Y$, so the question arises: Under what conditions will it result in an efficient point in the transformation set of the state that occurs? That is, if $y$ is efficient in $Y$, when will it follow that $(y^1, y^{2j})$ is efficient in $Y_j$? The geometry of this question is completely straightforward. The local condition is bafflingly trival. $y$ efficient in $Y$ is efficient in $Y_j$ if and only if $(y^1, y^{2j})$ is efficient in $Y_j$. Global statements are slightly more interesting.

THEOREM 4: *Let* j$\epsilon$S. *If every* y *efficient in* Y *is* j-*efficient, then* Y *is weakly independent with respect to* j.

*Proof*: Suppose not. Then the absence of weak independence with respect to $j$ implies the following.

*Case* 1: For some $w\epsilon W_j$ and some $y\epsilon Y$ such that $y^1=w^1$, there is no $z\epsilon Y$ such that $z^1=y^1=w^1$, $z^{2i}=y^{2i}$, $i\epsilon S-\{j\}$, and $(z^1, z^{2j})\epsilon W_j$. Choose $v$ efficient, $v\epsilon Y$ such that $v^1=y^1$, $v^{2i}=y^{2i}$, $i\epsilon S-\{\mathrm{j}\}$. Then $(v^1, v^{2j})$ is not an element of $W_j$. But this contradicts the hypothesis, since $v$ is efficient in $Y$ but not $j$-efficient. The contradiction shows that weak independence with respect to $j$ must hold.

*Case* 2: For $w\epsilon W_j$, $y\epsilon Y$ such that $w^2=y^{2j}$, the argument is entirely similar.

Q.E.D.

COROLLARY 4.1: *If every* y *efficient in* Y *is* j-*efficient for all* j$\epsilon$S, *then* Y *is weakly independent with respect to all* j$\epsilon$S.

THEOREM 5: *Let* Y *be weakly independent with respect to given* j$\epsilon$S. *Then any* y *efficient in* Y *is* j-*efficient.*

*Proof*: Let $y\epsilon Y$, $y$ efficient. Suppose $y$ is not $j$-efficient, then the following holds:

*Case 1*: There is $w\epsilon W_j$, $w^1=y^1$ such that $(y^1, y^{2j})$ is not an element of $W_j$. But by weak independence there is $z\epsilon W_j$ such that $(y^1, y^{21}, \ldots, y^{2j-1}, z^2, y^{2j+1}, \ldots)\epsilon Y$. Thus $y$ was not efficient to start with.

*Case 2*: $w^2=y^{2j}$ is entirely similar. The contradiction proves the theorem.

Q.E.D.

Theorems 4 and 5 give necessary and sufficient global conditions for ex ante efficiency to imply ex post efficiency. They imply that if an economy's ex ante transformation set is weakly independent with respect to $j$, knowledge as to whether state $j$ will occur will not enhance productive efficiency. Efficiency considerations provide no reason to investigate whether state $j$ will occur. Conversely, if one knows that the ex ante transformation set is not weakly independent with respect to $j$, there is reason to believe that productive efficiency will be enhanced by knowledge of whether state $j$ will occur.

## VII. Economies Where the Necessary Conditions for Arrow Optimum to Imply Ex Post Pareto Optimum Are Sufficient

The ex post trade-offs between present and future goods in production and consumption depend on the subjective probabilities on the basis of which ex ante decisions are made. Thus, the necessary conditions above will not in general be sufficient to guarantee ex post Pareto optimum. There are special cases where they are sufficient.

THEOREM 6: *Let* Y *be intratemporal. Let* $x^t$, $t \epsilon T$, *be an Arrow optimum. Let state* $j^*$ *occur. Then if subjective probabilities for* $j^*$ *are effectively similar* $(x^t, x^{t2j^*})$, $t \epsilon T$, *is an ex post Pareto optimum.*

*Proof*: It is sufficient to show that there is $p^* \epsilon \Omega_P$, $p^* >> 0$ supporting the allocation on production and consumption sides.[9] By effective similarity there is $(k, p)$ such that $p$ supports $\sum_{t \epsilon T} (x^{t1}, x^{t2j})$ on the production side. Let $p^* = \left(p^1, \frac{1}{k} p^2\right)$.

Intratemporality implies that $(ap^1, bp^2)$ supports $(x^{t1}, x^{t2j^*})$ for any $a, b > 0$. On the consumption side, consider $z = (z^1, z^{2j^*})$ so that $p^* \cdot \leqslant 0$. Then $(p^1, p^2) \cdot (kz^1, z^{2j^*}) \leqslant 0$. Dropping a few superscripts for convenience, we have

$$[v_2(x^{2j^*} + z^{2j^*}) - v_2(x^{2j^*})]k \leqslant v_1(x^1) - v_1(x^1 + kz^1)$$

by concavity of $v_1$

$$\leqslant v_1(x^1) - (1-k)v_1(x^1) - kv_1(x^1 + z^1)$$
$$= k[v_1(x^1) - v_1(x^1 + z^1)].$$

So

$$v_2{}^{tj^*}(x^{t2j^*} + z^{2j^*}) - v_2{}^{tj^*}(x^{2tj^*}) \leqslant v_1{}^t(x^{t1}) - v_1{}^t(x^{t1} + z^1)$$

for any $z$ so that $p^* \cdot z \leqslant 0$. Thus $u^{tj^*}(x^t + z) \leqslant u^{tj^*}(x^t)$ for all $z$ such that $p^* \cdot z \leqslant 0$. Thus $x^t$ maximizes $u^{tj^*}(w)$ for all $w$ so that $p^* \cdot w \leqslant p^* \cdot x^t$. Thus $p^*$ supports $x^t$.

Q.E.D.

COROLLARY 6.1: *In a pure exchange economy universal similarity of subjective probabilities for the state that occurs is a necessary*

9. By Arrow, "An Extension of the Basic Theorems," *op. cit.*

*and sufficient condition for an Arrow optimum to be an ex post Pareto optimum.*

*Proof of Corollary*: In a pure exchange economy universal similarity and effective similarity are equivalent, since any $p >> 0$ supports the (unique) output on the production side.

Q.E.D.

If an economy has intratemporal production, effective similarity of subjective probabilities ensures ex post Pareto optimum. Thus, at least for these economies, information telling what state of the world will occur is not particularly important for the achievement of ex post Pareto optimum. Pareto optimum results not from the accuracy of traders' beliefs but from their unanimity. Theorems 3 and 6 come very close to establishing effective similarity as a necessary and sufficient condition in an intratemporal economy for Arrow optimum to result in ex post Pareto optimum. Necessity does not hold in those instances (comparatively rare in an intratemporal economy) where effective and universal similarity do not coincide.

## Table of Notation

The definitions here are heuristic. Precise definitions are in context.

| | |
|---|---|
| $f(t, j)$ | trader $t$'s subjective probability of state $j$ occurring in period 2 |
| $j$ | possible state of the world in period 2 |
| $j^*$ | state of the world that actually obtains in period 2 |
| $K(t, x^t, j)$ | set of prices and subjective probabilities of state $j$ consistent with $t$ holding $x^t$ |
| $p$ | price vector $2n$ or $n(\lvert S\rvert+1)$ dimensional |
| $(p^1, p^{2j*})$ | $2n$ dimensional vector consisting of those components of $p$ representing period 1 goods and period 2 contracts for contingency $j^*$ |
| $p^*$ | $2n$ dimensional vector representing a price system that, if the distribution arrived at in the market for goods and contingent contracts were Pareto optimal, would support that distribution, were a market to be reestablished once it was known that $j^*$ obtains |
| $S$ | set of conceivable states of the world in period 2 |
| $T$ | set of all traders |
| $t$ | trader, element of $T$ |
| $u^{tj}$ | $t$'s utility function in the event state $j$ occurs |
| $v_1{}^t$ | if $u^{tj}$ separable over time, the period 1 part |
| $v_2{}^{tj}$ | if $u^{tj}$ separable over time, the period 2 part in the event $j$ occurs |
| $W_j$ | set of efficient points in $Y_j$ |

| | |
|---|---|
| $x, z$ | commodity-contingent contract purchase vector; $2n$ or $(\|S\|+1)n$ dimensional |
| $x^1$ | period 1 consumption components of $x$ |
| $x^{2j*}$ | period 2 contingency $j^*$ components of $x$ |
| $Y$ | ex ante transformation set; $(\|S\|+1)n$ dimensional |
| $Y_j$ | intertemporal transformation set in case event $j$ holds in period 2; $2n$ dimensional |
| $y$ | element of $Y$ or $Y_j$ |
| $\Omega_A$ | nonnegative orthant of $E^{(\|S\|+1)n}$ |
| $\Omega_P$ | nonnegative orthant of $E^{2n}$ |

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