Responding to Martin Hellwig's Presidential Address: Smith, Knapp, Menger, Hicks, and Tobin Deal with the Hahn Problem[1]

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"The Hahn problem: Problem 0. Why does fiat money have a positive value in exchange against goods and services even though it is not intrinsically useful?"

—— Martin Hellwig (1993)

Abstract

The "Hahn Problem" is to demonstrate in a Walrasian general equilibrium model the positive value of fiat money. In a trading post model of N commodities there are $\frac{1}{2}N(N-1)$ commodity-pairwise trading posts. Taxation — and fiat money's guaranteed value in payment of taxes — explains the positive equilibrium value of fiat money. A bid/ask spread at each trading post reflects transaction costs incurred at the post. The large size of government purchases paid for in fiat money interacts with scale economies in transaction cost to make fiat money the low-transaction-cost commodity. Thus fiat money becomes the unique actively used medium of exchange in general equilibrium. Fiat money equilibrium is characterized by all transactions concentrated on the trading posts trading the N commodities for fiat money; the remaining barter trading posts are priced but inactive in equilibrium.

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1 The Hahn Problem

Frank Hahn wrote a variety of fundamental contributions to Walrasian general equilibrium theory and monetary theory. But he came to a pessimistic interim conclusion, Hahn (1982), "The...challenge that...money poses to the theorist is this: the best developed model of the economy cannot find room for it. The best developed model is, of course, the Arrow-Debreu version of a Walrasian general equilibrium."

A generation ago, Prof. Martin Hellwig(1993) restated this concern as a research agenda and a challenge to economic theory:

> "To obtain a satisfactory framework for monetary economics, we must abandon the Walrasian approach with its assumption that in any period all agents trade simultaneously and multilaterally with each other. We must replace the Walrasian multilateral exchanges by a decentralized system of multiple bilateral exchanges in which there is a need for a medium of exchange and the role of money in transactions can be made explicit."

Hellwig summarizes a family of concerns as forms of the "Hahn Problem" based on Hahn (1965):

> "Problem 0. Why does fiat money have a positive value in exchange against goods and services even though it is not intrinsically useful?
>
> "Problem 1. Why does worthless fiat money have a positive value in exchange against goods and services when there are other assets whose own rates of return in each period exceed the own rate of return on money?
>
> "Problem 4. How can the theory of the transactions demand for money be integrated into an analysis of market equilibrium? "

This essay responds to Prof. Hellwig's challenge. The plan is to formulate an example, following the Walras-Arrow-Debreu model, that derives a positive equilibrium price for government-issued fiat money as the result of elementary price theory. Further, fiat money will be the universal common medium of exchange. The properties of money are conclusions following from cost and pricing, not assumptions. The model has a single time period, so intertemporal asset properties (the focus of Hellwig's Problems 2 and 3 [2]) will not be

[2] Problem 2: Why should cash-in-advance constraints be imposed? Problem 3. What is the relation between stocks and flows in a monetary economy?

developed. Nevertheless, there is active bilateral exchange and price determination. The positive value and transactions role of fiat money are outcomes of the equilibrium. Positive equilibrium value of fiat money and its role in exchange are results, not an assumption. Following Hellwig's suggestion, the essential step is to decompose the Walras-Arrow-Debreu market model from a single grand exchange into many smaller transactions, each with its own budget constraint. An endogenous outcome then is that some goods or instruments will be carriers of value moving between transactions. Fiat money is determined to be the unique carrier of value between exchanges as a result of elementary properties.

1.1 Outline of the model

The treatment here presents a trading post model of N commodities with $\frac{1}{2}N(N-1)$ commodity pairwise trading posts. Trading post transaction costs are reflected in the spread between bid and ask prices. Transaction costs display scale economies — high volume markets are low transaction cost markets. Government with taxing power accepts tax payments in the government-issued (otherwise useless) fiat money, ensuring a positive value for fiat money. Government (a large economic agent) spends its fiat money tax receipts at trading posts where fiat money exchanges for goods. High volume at those trading posts generates low transaction costs, narrow bid/ask spreads. But narrow bid/ask spreads lead all economic agents to use the fiat money trading posts. Government's taxing power, the large size of government purchases (for fiat money), and scale economies in transaction costs make fiat money the common medium of exchange.

1.2 The Most Saleable Good

The most elementary function of money — the medium of exchange — is as a carrier of value held between successive transactions. Carl Menger (1892) reminds us that the distinguishing feature of the medium of exchange should be liquidity. A simple characterization of liquidity is the difference between the bid price and the ask price. A commodity that acts as a medium of exchange is necessarily repeatedly bought (accepted in trade) and sold (delivered in trade). Therefore an instrument with a narrow spread between bid and ask price is priced to encourage households to use it as a carrier of value between trades, as a medium of exchange with relatively low cost.

2 The Trading Post Model

2.1 Trading Posts

The trading post model consists of N commodities traded pairwise at $\frac{1}{2}N(N-1)$ trading posts with distinct bid and ask prices. The bid/ask spread reflects transaction costs. Walras (1874) forms the picture this way (assuming m distinct commodities):

> "we shall imagine that the place which serves as a market for the exchange of all the commodities... for one another is divided into as many sectors as there are pairs of commodities exchanged. We should then have $\frac{\mathrm{m(m-1)}}{2}$ special markets each identified by a signboard indicating the names of the two commodities exchanged there as well as their ... rates of exchange..."

The trading post model decomposes the trading plans of each household into many separate transactions. The pattern of active trade is endogenously determined as part of the equilibrium of the trading post economy. The general equilibrium (of an example) with a common medium of exchange is developed below in sections 4 and 5; equilibrium bid prices appear in Table 1. Households create trading plans to optimize utility subject to prevailing prices and subject to a budget constraint at each post. A barter equilibrium occurs when most trading posts are active in equilibrium — most goods trading directly for most other goods. A monetary equilibrium occurs if active trade is concentrated on a few trading posts, those trading the common medium of exchange against most other goods.

Augment the non-monetary Arrow-Debreu model with two additional structures sufficient to give endogenous monetization in equilibrium: multiple budget constraints (one at each transaction, not just on net trade) and transaction costs. The choice of which trading posts a typical household will trade at is part of the household optimization. The equilibrium structure of exchange is the array of trading posts that actually host active trade. The determination of which trading posts are active in equilibrium is endogenous and characterizes the monetary character of trade. The equilibrium is monetary with a unique money if only N trading posts out of $\frac{1}{2}N(N-1)$ are active, those trading all goods against 'money.'

Let there be N commodities, numbered $1, 2, ..., N$. Goods are are traded in pairs — good i for good j — at specialized trading posts. The trading post for trade of good i versus good j (and vice versa) is designated $\{i,\ j\}$; trading post $\{i{,}j\}$ is the same trading post as $\{j,\ i\}$. Trading post $\{i,\ j\}$ actively buys and (re)sells both i and j. Trade as a resource using activity is modeled by describing the post's transaction costs. Transaction costs are visible to the households as the bid/ask spread between commodity prices at each trading post. The trading post itself operates at zero profit. $C^{\{i,j\}}$ is trading post {i,j}'s transaction cost.

2.2 Transactions and Prices

Let $b_\ell^{h,\{i,j\}}$ = purchase of good ℓ by household h at trading post $\{i, j\}$, and let $s_\ell^{h,\{i,j\}}$ = sale of good ℓ by household h, at trading post $\{i, j\}$.

The bid prices (the prices at which the trading post will buy from households) at $\{i, j\}$ are $q_i^{\{i,j\}}$, $q_j^{\{i,j\}}$ for goods i and j respectively. The price of i is in units of j. The price of j is in units of i. The ask price (the price at which the trading post will sell to households) of j is the inverse of the bid price of i (and vice versa). That is, $(q_i^{\{i,j\}})^{-1}$ and $(q_j^{\{i,j\}})^{-1}$ are the ask prices of j and i respectively at $\{i, j\}$. The trading post $\{i, j\}$ covers its costs by the difference between the bid and ask prices of i and j, that is, by the spread $(q_j^{\{i,j\}})^{-1} - q_i^{\{i,j\}}$ and the spread $(q_i^{\{i,j\}})^{-1} - q_j^{\{i,j\}}$. Transaction costs at the trading post are incurred in goods i and j, acquired in trade and defrayed through the difference in bid and ask prices.

Given $q_i^{\{i,j\}}, q_j^{\{i,j\}}$, for all $\{i, j\}$, household h then forms its buying and selling plans, deciding which trading posts to use to execute its desired trades.

2.3 Budget and Nonnegativity Constraints

A typical household h, has an endowment $r^h \in R_+^N; r_n^h$ is h's endowment of good n. h formulates its trading plan subject to the following budget and nonnegativity constraints:

(T.i) $b_n^{h\{i,j\}} > 0$ only if $n = i, j$; $s_n^{h\{i,j\}} > 0$ only if $n = i, j$.

(T.ii) $b_i^{h\{i,j\}} \leq q_j^{\{i,j\}} \cdot s_j^{h\{i,j\}}$, $\quad b_j^{h\{i,j\}} \leq q_i^{\{i,j\}} \cdot s_i^{h\{i,j\}}$ for each $\{i, j\}$.

(T.iii) $x_n^h = r_n^h + \sum_{\{i,j\}} b_n^{h\{i,j\}} - \sum_{\{i,j\}} s_n^{h\{i,j\}} \geq 0,\ \ 1 \leq n \leq N$.

Note that condition (T.ii) defines a budget balance requirement at the transaction level, implying the decentralized character of trade, consistent with the recommendation in Hellwig's presidential address. Since the budget constraint applies to each pairwise transaction separately, there may be a demand for a carrier of value to move purchasing power between distinct transactions. h faces the array of bid prices $q_i^{\{i,j\}}, q_j^{\{i,j\}}$ and chooses $s_n^{h\{i,j\}}$ and $b_n^{h\{i,j\}}$, $n = i, j,\ i, j = 1, 2, \ldots, N, i \neq j$, to maximize $u^h(x^h)$ subject to (T.i), (T.ii), (T.iii). That is, h chooses which pairwise markets to transact in and a transaction plan to optimize utility, subject to a multiplicity of pairwise budget constraints.

2.4 Competitive Equilibrium

$y_i^{\{i,j\}B}$ denotes trading post {i,j}'s purchase of i. $y_i^{\{i,j\}S}$ denotes its sale of i.

A **market equilibrium** consists of $q_i^{o\{i,j\}}$, $q_j^{o\{i,j\}}$, $1 \leq i, j \leq N$, so that :

- For each household h, there is a utility optimizing plan $b_n^{oh\{i,j\}}$, $s_n^{oh\{i,j\}}$, (subject to T.i, T.ii, T.iii) so that $\sum_h b_n^{oh\{i,j\}} = y_n^{o\{i,j\}S}$, $\sum_h s_n^{oh\{i,j\}} = y_n^{o\{i,j\}B}$, $n = i, j$, for each $\{i, j\}$, each n, where
- $y_n^{o\{i,j\}S} \leq y_n^{o\{i,j\}B}$, $n = i, j$.
- $y_i^{o\{i,j\}B} - y_i^{o\{i,j\}S} + y_j^{o\{i,j\}B} - y_j^{o\{i,j\}S} = C^{\{i,j\}}$ for all $1 \leq i, j \leq N, i \neq j$.

The expression in the last bullet is a zero profit condition.

The pattern of trade and trading post activity is an outcome of the market equilibrium. In equilibrium, it is possible that most of the $\frac{1}{2}N(N-1)$ trading posts be active. In that case, the equilibrium supports barter. Alternatively, if most trading posts are inactive and a narrow band of N posts is active, all trading a single common instrument versus N goods, then the equilibrium is monetary with the commonly traded instrument as the unique 'money.'

2.5 Fiat Money

In order to study fiat money we introduce a government with the unique power to issue fiat money. Fiat money is intrinsically worthless; it enters no one's utility function. But government is uniquely capable of declaring it acceptable in payment of taxes. Adam Smith (1776), Georg Knapp(1905, [1924]), and Abba Lerner (1947) remind us of taxation's role.

As an economic agent, government is denoted G. Government sells tax receipts, the $N{+}1^{\text{st}}$ good. It also sells good $N{+}2$, an intrinsically worthless instrument, (latent) fiat money, that government undertakes to accept in payment of taxes, that is, in exchange for $N+1$.

Government uses its revenue to purchase a variety of goods $n = 1, ..., N$, in the amount x_n^G.

Good $N{+}2$ represents latent fiat money. Government, G, sells $N{+}1$ (tax receipts) for $N{+}2$ at a fixed ratio of one-for-one. The trading post $\{N{+}1, N{+}2\}$ where tax receipts are traded for $N{+}2$ operates with zero transaction cost. The market clearing price of $N{+}2$ is to be determined in equilibrium. Its acceptability in payment of taxes will encourage positive value. If, in addition, $N{+}2$ trades at sufficiently low transaction cost, then it becomes the common medium of exchange.

3 Household Population, An Example

The population of trading households is denoted Θ. Households formulate their trading plans deciding how much of each good to trade at each pairwise trading post.

To generate an example of a monetary equilibrium in the trading post model, specify Θ more precisely. Let $N \geq 3$. Ω denotes the greatest integer $\leq (N-1)/2$.

Let [i,j] denote a household endowed with good i who prefers good j; $i \neq j$, i,j = 1, 2, ..., N. Household [i,j] 's endowment is $A > 0$ of commodity i. Denote the endowment of household [i,j] as $r_i^{[i,j]} = A$. The typical household $[i, j]$ in Θ desires to purchase tax receipts to the extent it prefers not to have a quarrel with the government's tax authorities. Government sets a target tax receipt purchase by the taxpayer of $\tau^{[i,j]}$. Household [i,j]'s utility function is

$$u^{[i,j]}(x) = x_j - 2[\max[(\tau^{[i,j]} - x_{N+1}^{[i,j]}), 0]] \tag{1}$$

That is, household $[i, j]$ values paying his taxes with a positive marginal utility up to his tax bill $\tau^{[i,j]}$ and with zero marginal utility for tax payments thereafter. Household [i,j] values consumption of good j only. He cares for i only as a resource to trade for j or to pay taxes. This is obviously an oversimplification — but it serves to focus the issue. Note that household [i,j] does not value good $N + 2$, the latent fiat money.

Let Θ be a population of households displaying a complete absence of double coincidence of wants, Jevons(1875). To deal with a manageable example, assume that there are Ω households endowed with each good and each household desires a good different from its endowment. There are Ω households endowed with good 1, preferring respectively, goods 2, 3, 4, ..., $\Omega + 1$: [1,2], [1,3], [1,4], ..., [1,$\Omega + 1$]. There are Ω households endowed with good 2, preferring respectively goods 3, 4, 5, ..., $\Omega + 2$: [2,3], [2,4], [2,5], ..., [2,$\Omega + 2$]. The roll call of households proceeds so forth, through [N, 1], [N, 2], [N, 3], ..., [N,Ω].

Envisage Θ's elements $[i, j]$ set round a clock-face at a position corresponding to the endowed good, i, eager to acquire j. j being 1, 2, ... , Ω, steps clockwise from i. For each household endowed with good i and desiring good j, [i,j], there is no precise mirror image, [j,i]. There is no scope for mutually satisfactory bilateral barter. Nevertheless, there are Ω households endowed with A of commodity 1 desiring a different good, and Ω households strongly preferring commodity 1 to all others. That is true for each commodity. Thus gross supplies equal gross demands, though there is no immediate opportunity for any two households to make a mutually advantageous trade. Jevons (1875) tells us that this is precisely the setting where money is suitable to facilitate trade and where barter is impractical.

4 Transaction Costs and Monetary Equilibrium

Let us characterize the transaction cost structure on commodity transactions and on tax payment transactions in the following way, emphasizing scale economies in transaction cost.

Defintion: *Scale economy transaction cost with taxation (SETCT)* The nonconvex (scale economy) transaction cost function is specified as

- For trading post $\{i,j\}$, $i,j = 1,2,\ldots,N,N+2$ is $C^{\{i,j\}} = \min[\delta^i y_i^{\{i,j\}B}, \gamma^i] + \min[\delta^j y_j^{\{i,j\}B}, \gamma^j]$ (Note that this specification includes trade of commodity $N+2$.)
- For trading post $\{N+1,j\}$, $j = 1,2,\ldots,N$, $C^{\{N+1,j\}} = [\delta^{N+1} y_{N+1}^{\{N+1,j\}B}] + [\delta^j y_j^{\{N+1,j\}B}]$, for $j = 1,2,\ldots,N$, where $\delta^{N+1} > \delta^i$, $i = 1,2,\ldots,N$, and
- For trading post $\{N+1,N+2\}$, $C^{\{N+1,N+2\}} = 0$.

where $\delta^i, \delta^j, \gamma^i, \gamma^j > 0$.

In words, the transaction technology looks like this: Trading post $\{i,j\}$ makes a market in goods i and j, buying each good in order to resell it. Transaction costs vary directly (in proportions δ^i, δ^j) with volume of trade at low volume and then hit a ceiling after which they do not increase with trading volume. The specification in (SETCT) is an extreme case: zero marginal transaction cost beyond the ceiling. Trading $N+2$ for $N+1$ is transaction costless. Trading other goods for $N+1$ (paying taxes in kind, rather than in fiat money) has a high transaction cost without scale economy.

Proposition 1: Let the population of households be Θ. Let $u^{[i,j]}$ be described by (1). Let $\tau^0 > 0$ be a constant. Let $0 < \tau^{[i,j]} = \tau^0 < A(1-\delta^{N+2})(1-\delta^i)$, all $[i,j] \in \Theta$. Let $x_n^G = \Omega\tau^0 q_{N+2}^{\{N+2,n\}}$ and $\delta^n > \gamma^n/\Omega\tau^0$ all $n = 1,2,...N,N+2$. Let $C^{\{i,j\}}$ be SETCT. Then there is an average cost pricing monetary equilibrium with good $N+2$ as the unique common medium of exchange.

The fiat money equilibrium price array is displayed in Table 1.

Demonstration of Proposition 1 : For $n,m \neq N+2$, set $q_n^{\{m,n\}} = (1-\delta^n)(1-\delta^m)$. $q_n^{\{N+2,n\}} = 1, q_{N+2}^{\{N+2,n\}} = (1-\frac{\gamma^n}{\Omega A})(1-\frac{\gamma^{N+2}}{\Omega A})$. Then all trade in $i = 1,2,\ldots,N$ goes through trading posts $\{i,N+2\}$ as the low cost venue.

The inequality $\delta^n > \gamma^n/\Omega\tau^0$ says that government is a big economic agent. The denominator, $\Omega\tau^0$, represents tax receipts from households endowed with each of the N commodities. Then government expenditure purchasing each of N goods is enough to induce scale economy in transaction cost.

The value of $q_{N+2}^{\{N+2,n\}}$ reflects the scale economy. The denominators in the term $(1-\frac{\gamma^n}{\Omega A})(1-\frac{\gamma^{N+2}}{\Omega A})$ show that a large number, 2Ω, of households are transacting at trading post $\{N+2, n\}$. Ω households endowed with n sell all of their endowment into the post in exchange for good $N+2$ and there is an equal volume of purchases (net of transaction cost) acquiring n for $N+2$. That high volume generates low average transaction cost as the costs γ^i, γ^j are spread over a large volume. Transaction costs are absorbed in the real commodities $n = 1, 2, \ldots, N$; $N+2$ is priced to pass through trade undiminished by transaction costs.

As is common in the setting of scale economy, there are multiple equilibria. Though $N+2$ is the common medium of exchange in this example (as demonstrated in Proposition 1) that is not guaranteed. Any commonly traded good, with trading volume sufficiently high to activate the scale economy, could become the common medium of exchange. Why should that be $N+2$? Since government is a large economic agent, active at high volume in many goods markets, government-issued $N+2$ is a high volume instrument, generating scale economies in transaction costs. Then in a dynamic adjustment, the economy will approach an allocation where $N+2$ is the common medium of exchange. That is the evolution developed next, in section 5.

5 Converging to Monetary Equilibrium

5.1 Tatonnement adjustment process for average cost pricing equilibrium

In a decentralized competitive market, will the economy converge on a single common medium of exchange? The scale economy in transaction cost in (SETCT) means that there may be a natural monopoly in the medium of exchange. The example below suggests that a dynamic adjustment will converge on that monopoly arrangement.

Prices will be adjusted by an average cost pricing auctioneer. Specify the following adjustment process for prices.

> STEP 0: Trading post $\{i, j\}$ prices only goods i, j. Other prices are unspecified there, indicating no available trade. At each pairwise

trading post the bid-ask spread is set to equal average cost at low trading volume.

CYCLE 1

STEP 1: Households compute their desired trades at the posted prices and report them for each pairwise market

STEP 2: Average costs (and average cost prices) are computed for each pairwise market based on the outcome of STEP 1. Prices are adjusted upward for goods in excess demand at a trading post, downward for goods in excess supply, with the bid-ask spread adjusted to average cost. CYCLE 2 Repeat STEP 1 (at the new posted prices) and STEP 2. CYCLE 3, CYCLE 4, ... repeat until the process converges and trading posts clear.

The plausible adjustment process above explains why government-issued fiat money becomes the unique common medium of exchange —- and would do so even in the absence of legal tender rules. Government has two distinctive characteristics: it has the power to support the value of fiat money by making it acceptable in payment of taxes; it is a large economic presence undertaking a high volume of transactions in the economy. Hence, government can make its fiat money the common medium of exchange merely by using it in government transactions. The scale economies implied will make fiat money the low transaction cost instrument and hence the most suitable medium of exchange, not just for government but for all transactors.

5.2 Convergence

Proposition 2: Let the population of households be Θ. Let $u^{[i,j]}$ be described by (1). Let $\tau^0 > 0$ be a constant. Let $0 < \tau^{[i,j]} = \tau^0 < A(1-\delta^{N+2})(1-\delta^i)$, all $[i,j] \in \Theta$. Let $x_n^G = \Omega\tau^0 q_{N+2}^{\{N+2,n\}}$ all $n = 1, 2, ...N$. Let $C^{\{i,j\}}$ be described by (SETCT). Let $(\gamma^{N+2}/\Omega\tau^0) < \delta^i$ all $i = 1, 2, ...N$. Then there exists a monetary average cost pricing equilibrium with taxation with good N+2 as the unique 'money.' That monetary equilibrium is the unique limit point of the tatonnement adjustment.

Demonstration of Proposition 2: Existence of the monetary equilibrium is demonstrated in Proposition 1 above. Convergence is argued below. Let the notation $n \oplus \ell$ represent $n + \ell$ mod N.

Step 0: For all goods, $n, m = 1, 2, \ldots, N, N+1, N+2$, $n \neq m$ set $q_n^{\{m,n\}} = (1-\delta^n)$.

Cycle 1, Step 1:

- For $n = 1, 2, \ldots, N$, $\ell = 1, 2, ..., \Omega$, let $s_n^{[n,n\oplus\ell]\{n,n\oplus\ell\}} = A - (\tau^0/q_n^{\{N+2,n\}})$, $b_{n\oplus\ell}^{[n,n\oplus\ell]\{n,n\oplus\ell\}} = (A - (\tau^0/q_n^{\{N+2,n\}}))q_n^{\{n,n\oplus\ell\}}$, $s_{N+2}^{[n,n\oplus\ell]\{N+2,N+1\}} = \tau^0 = b_{N+1}^{[n,n\oplus\ell]\{N+2,n\}}$; $b_{N+2}^{[n,n\oplus\ell]\{N+2,n\}} = \tau^0$, $s_n^{[n,n\oplus\ell]\{N+2,n\}} = \tau^0/q_n^{\{N+2,n\}}$
- For $n = 1, 2, ..., N$, let $s_{N+2}^{G\{N+2,n\}} = \Omega\tau^0$, $b_n^{G\{N+2,n\}} = \Omega\tau^0 q_{N+2}^{\{N+2,n\}}$.

Cycle 1, Step 2:

- For $n, m \neq N+2$, $n \neq m$, set $q_n^{\{m,n\}} = (1-\delta^n)$. $\min[\delta^n, \gamma^n/\Omega\tau^0] = \gamma^n/\Omega\tau^0$. Thus $q_n^{\{N+2,n\}} = (1-\gamma^n/\Omega\tau^0)(1-\gamma^{N+2}/\Omega\tau^0)$, $q_{N+2}^{\{N+2,n\}} = 1$.

Cycle 2, Step 1:

- For $n = 1, 2, ...N$, let $s_{N+2}^{G\{N+2,n\}} = \Omega\tau^0$, $b_n^{G\{N+2,n\}} = \Omega\tau^0 q_{N+2}^{\{N+2,n\}}$; $s_{N+1}^{G\{N+1,N+2\}} = N\Omega\tau^0$, $b_{N+2}^{G\{N+1,N+2\}} = N\Omega\tau^0$; $b_{N+1}^{[n,n\oplus\ell]\{N+2,N+1\}} = \tau^0$, $s_{N+2}^{[n,n\oplus\ell]\{N+2,N+1\}} = \tau^0$; $s_n^{[n,n\oplus\ell]\{N+2,n\}} = A$, $b_{N+2}^{[n,n\oplus\ell]\{n,N+2\}} = Aq_n^{\{N+2,n\}}$; $s_{N+2}^{[n,n\oplus\ell]\{n\oplus\ell,N+2\}} = Aq_n^{\{N+2,n\}} - \tau^0$, $b_{n\oplus\ell}^{[n,n\oplus\ell]\{n\oplus\ell,N+2\}} = (Aq_n^{\{N+2,n\}} - \tau^0)q_{N+2}^{\{n\oplus\ell,N+2\}}$.

Cycle 2, Step 2:

- For $n, m \neq N+2$, set $q_n^{\{m,n\}} = (1-\delta^n)$.
  $q_n^{\{N+2,n\}} = (1-\min[\delta^n, \gamma^n/\Omega A])(1-\gamma^{N+2}/\Omega A)$, $q_{N+2}^{\{N+2,n\}} = 1$.

Cycle 3, Step 1: Repeat Cycle 2, Step 1.

Cycle 3, Step 2: Repeat Cycle 2, Step 2.

Convergence. Pricing has converged to the equilibrium of Proposition 1. $N+2$ is the common medium of exchange.

What's happening in Proposition 2? Scale economies are taking their course! Government expenditures in all goods markets in exchange for N+2 (and large household demand to acquire N+2 to finance tax payments) result in a large trading volume on the trading posts for good N+2 versus $n = 1, ..., N$. Volume is large enough that scale economies kick in. The average cost pricing auctioneer adjusts prices, the bid/ask spread, to reflect the scale economies. The bid/ask spreads incurred on trading m for $m \oplus \ell$, where $m =$

$1, 2, \ldots, N$, by trading through the intermediary $N{+}2$ become considerably narrower than on trading m for $m \oplus \ell$ directly. The price system then directs each household to the market $\{m, N{+}2\}$ where its endowment is traded against good $N{+}2$. The household sells all its endowment there for $N{+}2$ and trades $N{+}2$ subsequently for tax payments and desired consumption. Scale economy has turned $N{+}2$ from a mere tax payment coupon into 'money,' the unique universally used common medium of exchange.

6 18th, 19th, 20th Century Solutions

6.1 Insights from Adam Smith, Georg Knapp, and James Tobin

A government-issued fiat instrument needs two properties to become the generally recognized 'money' : positive equilibrium price, general use as the principal carrier of value in exchange. These attributes can be derived from more elementary properties.

Positive equilibrium value of the fiat money can be conveyed by acceptability in payment of taxes. And positive equilibrium value will allow government to purchase its needs by paying for purchases in fiat money. Scale economy in transaction cost and the large scale imparted by government transactions generate liquidity of fiat money. Scale economy leads to natural monopoly — uniqueness of the common medium of exchange. Liquidity shows up as low marginal and average transaction cost, narrow bid/ask spreads on transactions including fiat money. Liquidity of fiat money leads to its general use as a carrier of value in transactions. These notions reflect ideas in the non-mathematical monetary theory literature, some as old as Adam (Smith), others going back a generation.

On taxation and the value of fiat money:

Adam Smith (1776) notes

> "A prince, who should enact that a certain proportion of his taxes be paid in a paper money of a certain kind, might thereby give a certain value to this paper money. "(v. I, book II, ch. 2).

G. F. Knapp's *Staatliche Theorie des Geldes* (1905, [1924]) says

> "Money is a creature of law...First and foremost, money frees us from our debts toward the state; for the state in emitting it, acknowledges that, in receiving, it will accept this means of payment."

Abba Lerner (1947) comments,

> "The modern state can make anything it chooses generally acceptable as money and thus establish its value quite apart from any connection, even of the most formal kind, with gold or with backing of any kind. It is true that a simple declaration that such and such is money will not do, even if backed by the most convincing constitutional evidence of the state's absolute sovereignty. But if the state is willing to accept the proposed money in payment of taxes and other obligations to itself the trick is done ... On the other hand if the state should decline to accept some kind of money in payment of obligations to itself, it is difficult to believe that it would retain much of its general acceptability."

Taxation — and fiat money's guaranteed value in payment of taxes — explains the positive equilibrium value of fiat money. That fiat money is legal tender is meaningless without a guarantee of the price at which it may be tendered. But acceptability in payment of taxes at a fixed rate creates a market-based value.

Government's large scale along with scale economies in transaction costs explain fiat money's uniqueness as the medium of exchange. Going back in the recesses of prehistory, Einzig(1966) writes:

> "Money tends to develop automatically out of barter, through the fact that favorite means of barter are apt to arise. "

On liquidity, transaction cost, and scale economy, James Tobin (1980, pp. 86-87) writes:

> "The use of a particular language or a particular money by one individual increases its value to other actual or potential users. Increasing returns to scale, in this sense, limits the number of languages or moneys in a society and indeed explains the tendency for one basic language or money to monopolize the field...

> "there is, as the language analogy suggests, arbitrariness and circularity in acceptability. Dollar bills and coins are acceptable because they are acceptable; of course, the state has a lot to do with making them acceptable, by defining them as acceptable for ... tax payments... Credible promises to pay those dollars, or to convey other such promises, also serve ... as widely acceptable media."

Helene Rey(2001), discussing her trading post model of foreign exchange markets, describes:

> "bilateral exchange markets whose efficiency depends on their 'thickness'. There is a strategic complementary in the exchange process ... 'thick market externality' "

6.2 Formalizing Smith, Knapp, and Tobin

The treatment above in sections 2 through 5 presents a bilateral trading post model, following Hellwig's suggestion. Government is introduced as a large economic agent, issuing a fiat instrument acceptable in payment of the government's taxes. Households desire to pay their taxes, ensuring a positive value for the fiat instrument (following Smith and Knapp). Transaction costs imply a spread between buying and selling (bid/ask spread) prices, Foley(1970). Scale economies in transaction costs (following Tobin and Rey) imply increased liquidity, lower average transaction cost with narrower bid/ask spread. That occurs in high volume trading posts where government makes its large purchases in exchange for its fiat instrument. Hence those trading posts using fiat money, with narrow bid/ask spread, are lowest cost to all transactors. Trading posts where the government fiat instrument trades against all other goods become the only trading posts in general use. As Menger (1892) tells us,

> "The theory of money necessarily presupposes a theory of the saleableness [*Absätzfahigkeit*, liquidity] of goods...[Call] goods...*more or less saleable*, according to the...facility with which they can be disposed of...at current purchasing prices or with less or more diminution... Men...exchange goods...for other goods...more saleable... [which] become generally acceptable media of exchange...
>
> "Men have been led...without legal compulsion...to exchange goods...for other goods...more saleable."

6.3 Hicks and Tobin Resolve Problem 1

Hellwig's statement of Hahn Problem 1 above, rate of return dominance by bills over money, was treated by Hicks (1935).

> "I think we have to look the frictions in the face, ... The most obvious sort of friction ... is the cost of transferring assets from one form to another. "

Tobin (with Golub) (1998) emphasizes the dual role of thick markets and transaction cost:

> "Ninety-day U.S. Treasury bills, payable to bearer, are not generally acceptable, and this fact can be rationalized in a number of ways. Bills can vary in price, and it takes ordinary folks a little time and some cost to sell them. They come only in large denominations. The trouble with the rationalization is that if Treasury bills were a medium of exchange they would not vary in price, and selling them would be quick and cheap-and also unnecessary. Even their indivisibility could be put up with, given that there are other media available for making change. The principal reason, then, that Treasury bills are not media of exchange is that they are not generally acceptable. This unsatisfactory circular conclusion underlines the essential point that general acceptability in exchange is one of those phenomena-like language, rules of the road, fashion in dress-where the fact of social consensus is much more important and much more predictable than the content."

Thus Hicks and Tobin between them tell us that it is an endogenous matter of transaction cost that differentiates interest-bearing default-free instruments payable in the common medium of exchange from that medium itself. Both can persist in positive quantities in household portfolios because the medium of exchange is needed for transactions and there are set-up costs in moving from that medium to its interest-bearing counterpart. See also Tobin(1956) and Baumol (1952). That is a good verbal explanation. Formalizing the argument in a general equilibrium model is a research agenda recommended by Hellwig, beyond the scope of this paper.

7 Conclusion: Government-Issued Fiat Money is a Natural Monopoly

7.1 Price Theory of Money

Fiat money is a puzzle in two dimensions: It is inherently worthless so why is it valuable? Why is it (and its close substitutes) the universal unique common medium of exchange? The answer to the first question is taxation payable in fiat money. The answer to the second is that scale economies in transaction costs make money a natural monopoly. Government's large scale secures the monopoly to government's fiat instrument.

7.2 Starting to Solve the Hahn Problems

The examples of Propositions 1 and 2 demonstrate a solution to Hahn Problem 0. Acceptability in payment of taxes assures the positive value of fiat money. High trading volume generates liquidity making fiat money the common medium of exchange.

The Propositions suggest but do not fully resolve Problems 1 and 4. Propositions 1 and 2 generate a demand for fiat money in transactions; that goes most of the way to solving Problem 4. But Problems 1 and 4 treat asset holding behavior, and asset yields, requiring a model with an explicit time dimension. But Propositions 1 and 2, along with the discussions of Hicks and Tobin tell us the result to expect. High trading volume designates fiat money as the common medium of exchange, and it will be used as a medium of exchange even if dominated in yield by other instruments. Then transaction costs between fiat money and higher yield instruments will determine the portfolio mix held by households in asset market equilibrium.

Hellwig stated the problem productively: "To obtain a satisfactory framework for monetary economics, we must ...replace the Walrasian multilateral exchanges by a decentralized system of multiple bilateral exchanges in which there is a need for a medium of exchange and the role of money in transactions can be made explicit." The general equilibrium example of sections 4 and 5 is a step in fulfilling Prof. Hellwig's plan.

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Table 1: Fiat Money Equilibrium Average Cost Bid Prices

selling:		**1**	**2**	**3**	**...**	**N**	**N+1**	**N+2**
buying:	**1**	X	$(1-\delta^1)(1-\delta^2)$	$(1-\delta^1)(1-\delta^3)$	$\cdots$	$(1-\delta^1)(1-\delta^N)$	$(1-\delta^1)(1-\delta^{N+1})$	$(1-\frac{\gamma^1}{\Omega A})(1-\frac{\gamma^{N+2}}{\Omega A})$
	2	$(1-\delta^1)(1-\delta^2)$	X	$(1-\delta^3)(1-\delta^2)$	$\cdots$	$(1-\delta^N)(1-\delta^2)$	$(1-\delta^2)(1-\delta^{N+1})$	$(1-\frac{\gamma^2}{\Omega A})(1-\frac{\gamma^{N+2}}{\Omega A})$
	3	$(1-\delta^3)(1-\delta^1)$	$(1-\delta^3)(1-\delta^2)$	X	$\cdots$	$(1-\delta^3)(1-\delta^N)$	$(1-\delta^3)(1-\delta^{N+1})$	$(1-\frac{\gamma^3}{\Omega A})(1-\frac{\gamma^{N+2}}{\Omega A})$
	...	$\cdots$	$\cdots$	$\cdots$	X	$\cdots$	$\cdots$	$\cdots$
	N	$(1-\delta^1)(1-\delta^N)$	$(1-\delta^2)(1-\delta^N)$	$(1-\delta^3)(1-\delta^N)$	$\cdots$	X	$(1-\delta^N)(1-\delta^{N+1})$	$(1-\frac{\gamma^N}{\Omega A})(1-\frac{\gamma^{N+2}}{\Omega A})$
	N+1	$(1-\delta^1)(1-\delta^{N+1})$	$(1-\delta^2)(1-\delta^{N+1})$	$(1-\delta^3)(1-\delta^{N+1})$	$\cdots$	$(1-\delta^N)(1-\delta^{N+1})$	X	1
	N+2	1	1	1	$\cdots$	1	1	X