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Shapley–Folkman theorem

The Shapley–Folkman theorem places an upper bound on the size of the non-convexities (loosely speaking, openings or holes) in a sum of non-convex sets in Euclidean N-dimensional space, R^N. The bound is based on the size of non-convexities in the sets summed and the dimension of the space. When the number of sets in the sum is large, the bound is independent of the number of sets summed, depending rather on N, the dimension of the space. Hence the size of the non-convexity in the sum becomes small as a proportion of the number of sets summed; the non-convexity per summand goes to zero as the number of summands becomes large. The Shapley–Folkman theorem can be viewed as a discrete counterpart to the Lyapunov theorem on non-atomic measures (Grodal, 2002).

The theorem is used to demonstrate the following properties:

- existence of approximate competitive general equilibrium in large finite economies with non-convex preferences (increasing marginal rate of substitution) or non-convex technology (bounded increasing returns; the U-shaped cost curve case);
- convergence of the core to the set of competitive equilibria (Arrow and Hahn, 1972; Anderson 1978).

It may also be used to characterize the solution of non-convex programming problems (Aubin and Ekeland, 1976).

For $S \subset R^N$, S compact, define rad(S), the radius of S, as a measure of the size of S. Define $r(S)$, the inner radius of S, and $\rho(S)$ inner distance of S, as measures of the non-convexity (size of holes) of S. Let con$S$ denote the closed convex hull of S (smallest closed convex set containing S as a subset).

$$\mathrm{rad}(S) \equiv \inf_{x \in R^N} \sup_{y \in S} |x - y|;$$
$$r(S) \equiv \sup_{x \in \mathrm{con} S} \inf_{\{T \subset S | T \text{spans} \ x\}} \mathrm{rad}(T);$$
$$\rho(S) \equiv \sup_{x \in \mathrm{con} S} \inf_{y \in S} |x - y|.$$

rad (S) is the radius of the smallest closed ball centred in con$S$ containing S. A set of points T is said to span a point x, if x can be expressed as a convex combination (weighted average) of elements of T. $r(S)$ is the smallest radius of a ball centred in the convex hull of S, so that the ball is certain to contain a set of points of S that span the ball's centre. Hence $r(S)$ represents a measure of breadth of non-convexities in S. $\rho(S)$ is the maximum distance from a point in con$S$ to (the nearest point of) S. Hence it represents the smaller of breadth or depth of non-convexities of S.

Let $S_1, S_2, \ldots, S_m$ be a (finite) family of m compact subsets of R^N. The vector sum of $S_1, S_2, \ldots, S_m$, denoted W is a set composed of representative elements of $S_1, S_2, \ldots, S_m$ summed together. W is defined as

$$W \equiv \sum_{i=1}^{m} S_i \equiv \left\{ w | w = \sum_{i=1}^{m} x^i, x^i \in S^i \right\}$$

where the sum in the brackets is taken over one element of each S_i.

Theorem (Shapley–Folkman): Let $S_1,\ldots, S_m$ be a family of m compact subsets of R^N; $W = \Sigma_{i=1}^m S_i$. Let $L \geq \text{rad}(S_i)$ for all S_i; let $n = \min(N, m)$. Then for any $x \in \text{con}\, W$

(i) $x = \Sigma_{i=1}^m x^i$, where $x^i \in \text{con} S_i$ and with at most N exceptions, $x^i \in S_i$;
(ii) there is $y \in W$ so that $|\mathrm{x} - \mathrm{y}| \leq L\sqrt{n}$.

Corollary (Starr): Let $S_1,\ldots, S_m$ be a finite family of compact subsets of R^N. $W = \Sigma_{i=1}^m S_i$. Let $L \geq r(S_i)$ for all S_i, $n = \min(m, N)$. Then for any $x \in \text{con}\, W$ there is $y \in W$ so that

$$|x - y| \leq L\sqrt{n}.$$

Corollary (Heller): Let $S_1,\ldots, S_m$ be a finite family of compact subsets of R^N; $W = \Sigma_{i=1}^m S_i$. Let $L \geq \rho(S_i)$ for all S_i, $n = \min(m, N)$. Then for any $x \in \text{con}\, W$ there is $y \in W$ so that

$$|x - y| \leq Ln.$$

Statements and proofs of the theorem and corollaries along with applications are available in Arrow and Hahn (1972) and Green and Heller (1981). Development of the theorem is due to L.S. Shapley and J.H. Folkman (private correspondence) with publication in Starr (1969). Extensions, alternative proofs, and applications appear in the other references.

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See also

< xref = xyyyyyy > perfect competition.

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Index terms

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