MIDTERM EXAMINATION — Suggested Answer to Questions 1

1. Let there be two commodities x,y in a pure exchange economy economy. The possible household consumption set is the nonnegative quadrant. $X^i \equiv \mathbf{R}^{\mathbf{2}}_{+}$. All households have the same preferences $\succeq_i$ characterized in the following way:
$(x^\circ, y^\circ) \succ_i (x', y')$ if $x^\circ + y^\circ > x' + y'$, OR if $x^\circ + y^\circ = x' + y'$ and $x^\circ > x'$.
$(x^\circ, y^\circ) \sim_i (x', y')$ if $(x^\circ, y^\circ) = (x', y')$.

(a) The preferences $\succeq_i$ do not fulfill C.V (Continuity) of Starr's *General Equilibrium Theory*. Give a mathematical demonstration of this property. A full proof is not required. What are the implications for demand behavior of the household?

(b) Assume the economy with household preferences $\succeq_i$ fulfills all of the assumptions of Starr's *General Equilibrium Theory*, draft second edition, Theorem 14.1, with the exception of C.V. In order to assure C.VII (Adequacy of Income), assume for all i, that $r^i \geq (2, 2)$ where the inequality holds co-ordinatewise.
In this economy, does there exist a competitive general equilibrium price vector? If 'yes' provide a demonstration or find an equilibrium price vector and equilibrium allocation (assume any additional properties of r^i needed to solve for the equilibrium allocation). If 'no' provide a demonstration. If the answer is 'possibly but not always,' explain fully. A full proof is not required.

Suggested Answer: (a) $A^i(x^\circ, y^\circ)$ is not a closed set. To demonstrate this consider a sequence of points superior to (x°, y°) in $A^i(x^\circ, y^\circ)$, $(x^\circ - 1 + 1/\nu, y^\circ + 1 + 1/\nu) \rightarrow (x^\circ - 1, y^\circ + 1)$. Each element of the sequence is in $A^i(x^\circ, y^\circ)$ but the limit point $(x^\circ - 1, y^\circ + 1)$ is inferior to (x°, y°) under the ordering $\succ_i$ and is not in $A^i(x^\circ, y^\circ)$.
The implication for household demand behavior is that preferences cannot be represented as a continuous utility function and that at some prices demand may respond discontinuously to price changes.

(b) No. There is no competitive equilibrium price vector. At any price where $p_x > p_y$ all demand is for y and demand for x is zero. At any price where $p_x \leq p_y$ all demand is for x and demand for y is zero. There is no market clearing price vector.

MIDTERM EXAMINATION — Suggested Answer Question 2

Restating the question: **2.** On the island of Vinopesce there are two perfectly divisible products: wine, y, and fish, x. The only factor of production is perfectly divisible labor, L. Maximum fish catch for the whole island is 100 fish. This is a static equilibrium problem: there are no conservation issues. There are ten perfectly competitive fishing firms, denoted j = 1, 2, ..., 10. Labor employed by firm j is denoted by L^j and by firm i (typically a dummy index) is L^i. All firms have the same technology

$$x^j = L^j, \text{ when } \sum_{i=1}^{10} L^i < 100$$

$$x^j = 100\frac{\mathrm{L^j}}{\sum_{\mathrm{i=1}}^{10}\mathrm{L^i}}, \text{ when } \sum_{i=1}^{10} L^i \geq 100$$

Firms behave myopically with regard to the congestion effect in fishing. Firm j treats $\sum_{i=1}^{10} L^i$ parametrically as Q, assuming

$$x^j = 100\frac{\mathrm{L^j}}{\mathrm{Q}}, \text{ when } Q \geq 100$$

Q denotes the total labor employed in fishing, treated parametrically by all firms (that is, firms do not recognize their own contribution to Q in optimizing their response to the Lindahl pricing scheme). Note that x^j is not differentiable with respect to Q at $Q = 100$, but for all values of $Q > 100$ we have $\frac{\partial x^j}{\partial Q} = -\frac{100}{Q^2}L^j$. Thus for $Q > 100, Q \approx 100, \frac{\partial x^j}{\partial Q} \approx -\frac{1}{100}L^j$.

Wine is produced under constant returns by many firms k, with the technology, $y^k = L^k$. There are 1000 laborers on Vinopesce, one per household, each endowed with one unit of (divisible) labor. All households have the same utility function

$$u^h(x^h, y^h) = x^h + .5y^h, \text{ where } x^h \text{denotes h's fish consumption, and } y^h$$

denotes h's wine consumption. Leisure is not valued. Households sell their labor at the competitive wage rate w. Set $p_y = 1$. That is, wine is the numeraire with price unity. The following quantities are determined in competitive equilibrium:

$$w = \text{competitive wage rate of labor} = 1$$

$$\sum_{i=1}^{10} x^i = \text{total fish harvest} = 100$$

$$\sum_{i=1}^{10} L^i = \text{total labor employed in fishing} = 200$$

$$p_x = \text{price of fish} = 2$$

Consequent wine output is 800. This competitive allocation is inefficient because of the congestion externality in fishing. A superior attainable allocation is 100 fish with 100 laborers employed in fishing and 900 wine. A Lindahl auctioneer proposes the following Lindahl pricing scheme to treat the externality:

> Q denotes the total labor employed in fishing, treated parametrically by all firms (that is, firms do not recognize their own contribution to Q in optimizing their response to the Lindahl pricing scheme). Each firm i, can specify a desired level of Q, Q^i, based on its own optimization and the Lindahl price of Q^i , t^i.
>
> $$t^j = \text{ firm j's Lindahl price of } Q,\ t = \sum_{i=1}^{10} t^i$$
>
> Firm j's Lindahl profit function is
>
> $\pi^j = p_x x^j - wL^j + t^j Q^j - tL^j$ where x^j, L^j, Q^j are j's decision variables.
>
> Q^j is firm j's selected level of Q given t^j. A Lindahl equilibrium occurs where there is market clearing in wine, fish, labor, and where $Q^j = Q = \sum_{i=1}^{10} L^i$ for all firms j. The Lindahl auctioneer recognizes that there is a variety of allocations of labor across firms consistent with a Lindahl equilibrium. He proposes $t^i = 0.1$, all i, as appropriate Lindahl pricing leading to an efficient allocation. The Lindahl auctioneer is thinking of an efficient allocation $L^i = 10$, all i.

Is there a Lindahl equilibrium at the value of t^i specified?

If not, explain why.

If so, find the allocation of labor across firms, and between wine and fish. Are firms optimizing their Lindahl profit functions? Is the allocation Pareto efficient? Explain fully, finding values for x^j, L^j, Q^j.

Suggested Answer: Yes, there is a Lindahl equilibrium at the suggested value of t^i. There are actually many Lindahl equilibria with various values of t^i but the symmetric treatment with the same t^i for all firms $i = 1, ..., 10$ seems simplest. For all firms $i = 1, 2, ..., 10$, consider the following values:

$L^i = 10$

$x^i = 10$
$Q^i = 100$

Note that $t^i = 0.1, i = 1, 2, \cdots, 10$ implies $t = 1$. As in the competitive equilibrium, assuming an interior solution, w = Lindahl equilibrium wage rate of labor = 1, p_x = price of fish = 2.

This results in $\pi^i = p_x x^i - wL^i + t^i Q^i - tL^i = 2 \cdot 10 - 1 \cdot 10 + 0.1 \cdot 100 - 1 \cdot 10 = 10$ for each firm i.

Why is this mix of values profit maximizing for the typical firm j? j's decision variables are L^j (with consequent x^j), and Q^j with consequent $\frac{\partial \pi^j}{\partial Q^j}$.

$$\frac{\partial \pi^j}{\partial L^j} = p_x \frac{\partial x^j}{\partial L^j} - w - t = 0, \text{at values specified above}$$

$$\frac{\partial \pi^j}{\partial Q^j} = t^j = 0.1 \text{ for } Q^j < 100$$

$$\frac{\partial \pi^j}{\partial Q^j} = p_x \frac{\partial x^j}{\partial Q^j} + t^j = 2(-\frac{1}{100} L^j) + .1 = -.1 \text{ for } Q^j > 100, Q^j \approx 100$$

so $Q^j = 100$ is an optimizing choice for j.

Assuming all firm ownership shares are uniformly distributed across households, each household's income = 1.1. Total fish harvest is 100, total labor to fishing is 100, total labor to wine is 900, total wine harvest is 900, total value of product is 1100 = total household income = 1.1×1000. Markets clear.

The allocation is Pareto efficient. It is technically efficient since it produces maximum attainable fish with minimal labor ($\sum L^i = 100$) consistent with that output and devotes all remaining labor to wine, the two valued goods. This is Pareto efficient inasmuch as the marginal fish product of labor (for $\sum L^i \leq 100$)is valued by household utility twice as much as the marginal wine product of labor.

Answers to 14.10, 14.11, 14.12

14.10. Show that each co-ordinate unit vector (1, 0, 0, ..., 0), (0, 1, 0, 0, ..., 0), etc. is a fixed point of Γ .

***Suggested Answer* :** Let the k^{th} co-ordinate unit vector be denoted e^k . Then

$$\Gamma_k(e^k) = \frac{1+0}{1+\sum_{n=1}^{N} 0} = 1$$

for k, and for $i \neq k$, we have

$$\Gamma_i(e^k) = \frac{0+0}{1+0} = 0$$

so e^k is a fixed point of Γ.

14.11. Let p* be a competitive equilibrium price vector. Show that p* is a fixed point of Γ, that is, $\Gamma(p^*) = p^*$.

Suggested Answer:

$$\Gamma_i(p^*) = \frac{p^*_i + |\min[p^*_i\ Z_i(p^*), 0]|}{1+\sum_{n=1}^{N} |\min[p^*_n\ Z_n(p^*), 0]|} = \frac{p^*_i + 0}{1+\sum_{n=1}^{N} 0} = p^*_i$$

14.12. Let p^o be a fixed point, $p^o = \Gamma(p^o)$. Is p^o is a competitive equilibrium price vector? [Hint: The question is not whether the economy has a competitive equilibrium or a fixed point of Γ. The question is whether a fixed point of this mapping is always a competitive equilibrium price vector.]

Suggested Answer: It is possible that p^o is a competitive equilibrium (as problem 2 demonstrates, but we cannot be sure that there is a competitive equilibrium merely because we have a fixed point of Γ . Problem 1 makes it clear that there are many fixed points that are not necessarily competitive equilibria.

15.1 Consider production without P.IV(b), but fulfilling P.I–P.III and P.IV(a). Formulate an example of Y^1 and Y^2 in $\mathbf{R}^2$ so that the set of points attainable in Y^1 is not bounded.

Suggested Answer: $Y^1 = \{(x, y)|x = -y; y \leq 0\}$, $Y^2 = \{(x, y)|y = -x; x \leq 0\}$, $r = (1, 1)$. Then the attainable subset of Y, where $Y = Y^1 + Y^2$, is bounded $= \{(x, y)|(-1, -1) \leq (x, y) \leq (1, 1), x + y = 0\}$, but the attainable subsets of Y^1 and of Y^2 are unbounded $= \{(x, y)|y \leq 0, x = -y\}$ and $= \{(x, y)|x \leq 0, y = -x\}$ respectively.